

SCALING OF HARMONIC OSCILLATOR EIGENFUNCTIONS AND THEIR NODAL SETS AROUND THE CAUSTIC

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ABSTRACT. We study the scaling asymptotics of the eigenspace projection kernels $\Pi_{\hbar,E}(x,y)$ of the isotropic Harmonic Oscillator $\hat{H}_{\hbar} = -\hbar^2\Delta + |x|^2$ of eigenvalue $E = \hbar(N + \frac{d}{2})$ in the semi-classical limit $\hbar \rightarrow 0$. The principal result is an explicit formula for the scaling asymptotics of $\Pi_{\hbar,E}(x,y)$ for x,y in a $\hbar^{2/3}$ neighborhood of the caustic \mathcal{C}_E as $\hbar \rightarrow 0$. The scaling asymptotics are applied to the distribution of nodal sets of Gaussian random eigenfunctions around the caustic as $\hbar \rightarrow 0$. In previous work we proved that the density of zeros of Gaussian random eigenfunctions of \hat{H}_{\hbar} have different orders in the Planck constant \hbar in the allowed and forbidden regions: In the allowed region the density is of order \hbar^{-1} while it is $\hbar^{-1/2}$ in the forbidden region. Our main result on nodal sets is that the density of zeros is of order $\hbar^{-\frac{2}{3}}$ in an $\hbar^{\frac{2}{3}}$ -tube around the caustic. This tube radius is the ‘critical radius’. For annuli of larger inner and outer radii \hbar^{α} with $0 < \alpha < \frac{2}{3}$ we obtain density results which interpolate between this critical radius result and our prior ones in the allowed and forbidden region. We also show that the Hausdorff $(d-2)$ -dimensional measure of the intersection of the nodal set with the caustic is of order $\hbar^{-\frac{2}{3}}$.

1. INTRODUCTION

This article is concerned with the scaling asymptotics of eigenspace projections of the isotropic Harmonic Oscillator

$$\hat{H}_{\hbar} = \sum_{j=1}^d \left(-\frac{\hbar^2}{2} \frac{\partial^2}{\partial x_j^2} + \frac{x_j^2}{2} \right), \quad (1)$$

and their applications to nodal sets of random Hermite eigenfunctions when $d \geq 2$. It is well-known that the spectrum of \hat{H}_{\hbar} consists of the eigenvalues

$$E = \hbar(N + d/2), \quad N \in \mathbb{Z}_{\geq 0}.$$

The semi-classical limit at the energy level $E > 0$ is the limit as $\hbar \rightarrow 0, N \rightarrow \infty$ with fixed E , so that \hbar only takes the values

$$\hbar_N := \frac{E}{N + d/2}.$$

We denote the corresponding eigenspaces by

$$V_{\hbar_N,E} := \{\psi \in L^2(\mathbb{R}^d) : \hat{H}_{\hbar_N}\psi = E\psi\}. \quad (2)$$

The eigenspace projections are the orthogonal projections

$$\Pi_{\hbar_N,E} : L^2(\mathbb{R}^d) \rightarrow V_{\hbar_N,E}. \quad (3)$$

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An important feature of eigenfunctions of Schrödinger operators $-\hbar^2\Delta + V$ is that as $\hbar \rightarrow 0$ and with fixed eigenvalue E , they are rapidly oscillating in the classically allowed region

$$\mathcal{A}_E := \{V(x) \leq E\},$$

and exponentially decaying in the classically forbidden region

$$\mathcal{F}_E := \mathcal{A}_E^c = \{V(x) > E\},$$

with an Airy type transition along the caustic

$$\mathcal{C}_E := \partial\mathcal{A}_E = \{V(x) = E\}.$$

This reflects the fact that a classical particle of energy E is confined to $\mathcal{A}_E = \{V(x) \leq E\}$. In the forbidden region, eigenfunctions exhibit exponential decay as $\hbar \rightarrow 0$, measured by the Agmon distance to the caustic. We refer to [Ag, HS] for background. In dimension one, eigenfunctions have no zeros in the forbidden region, but in dimensions $d \geq 2$ they do. In the allowed region, nodal sets of eigenfunctions behave in a similar way to nodal sets on Riemannian manifolds [Jin], but in the forbidden region they are sparser. The only results at present on forbidden nodal sets seem to be those of [HZZ, CT]. This article contains the first results on the behavior of nodal sets in the transition region around the caustic. The scaling asymptotics of zeros around the caustic is analogous in many ways to the scaling asymptotics of eigenvalues of $N \times N$ random Hermitian matrices around the edge of the Wigner distribution in [TW], and as will be seen, the scaled Airy kernel of [TW] is the same as the scaled eigenspace projections when $d = 3$ (see Remark 1).

When $d = 1$, the eigenspaces $V_{\hbar_N, E}$ have dimension 1 and it is a classical fact (based on WKB or ODE techniques) that Hermite functions and more general Schrödinger eigenfunctions exhibit Airy asymptotics at the caustic (turning points). See for instance [Sz, O, T, Th, FW]. The main purpose of this article is to formulate and prove a generalization of these Airy asymptotics to all dimensions for the isotropic Harmonic Oscillator. Instead of considering individual eigenfunctions, we consider the scaling asymptotics of the eigenspace projection kernels (3) with x, y in an $\hbar^{2/3}$ -tube around \mathcal{C}_E . Our main result gives scaling asymptotics for the eigenspace projection kernels (3) around a point $x_0 \in \mathcal{C}_E$ of the caustic. To state the result, we introduce some notation. Let x_0 be a point on the caustic $|x_0|^2 = 1$ for $E = 1/2$. Points in an $\hbar^{2/3}$ neighborhood of x_0 may be expressed as $x_0 + \hbar^{2/3}u$ with $u \in \mathbb{R}^d$. The caustic is a $(d-1)$ -sphere whose normal direction at x_0 is x_0 , so the normal component of u is u_1x_0 when $|x_0| = 1$, where $u_1 := \langle x_0, u \rangle$. We also put $u' := u - u_1x_0$ for the tangential component, and identify $T_{x_0}\mathcal{C}_E \cong T_{x_0}^*\mathcal{C}_E \cong \mathbb{R}^{d-1}$. By rotational symmetry, we may assume $x_0 = (1, 0, \dots, 0)$, so that $u = (u_1, u_2, \dots, u_d) =: (u_1; u')$.

THEOREM 1.1. *Let x_0 be a point on the caustic $|x_0|^2 = 1$ for $E = 1/2$. Then for $u, v \in \mathbb{R}^d$,*

$$\Pi_{\hbar, 1/2}(x_0 + \hbar^{2/3}u, x_0 + \hbar^{2/3}v) = \hbar^{-2d/3+1/3} \Pi_0(u, v)(1 + O(\hbar^{1/3})), \quad (4)$$

where

$$\Pi_0(u_1, u'; v_1, v') := 2^{2/3}(2\pi)^{-d+1} \int_{\mathbb{R}^{d-1}} e^{i\langle u' - v', p \rangle} \text{Ai}(2^{1/3}(u_1 + p^2/2)) \text{Ai}(2^{1/3}(v_1 + p^2/2)) dp, \quad (5)$$

and $u_1 := \langle x_0, u \rangle$, $u' := u - u_1 x_0$ (similarly for v_1 .) On the diagonal, let $|x|^2 = |x_0 + \hbar^{2/3} u|^2 = 1 + \hbar^{2/3} s + O(\hbar^{4/3})$ with $s = 2\langle x_0, u \rangle \in \mathbb{R}$. Then,

$$\Pi_{\hbar}(x, x) = 2^{-d+1} \pi^{-d/2} \hbar^{(1-2d)/3} \text{Ai}_{-d/2}(s) (1 + O(\hbar^{1/3})). \quad (6)$$

The error terms in (4) and (6) are uniform when u, v, s vary over a compact set.

Above, Ai is the Airy function, and $\text{Ai}_{-d/2}$ is a *weighted Airy function*, defined for $k \in \mathbb{R}$ by

$$\text{Ai}_k(s) := \int_{\mathcal{C}} T^k \exp\left(\frac{T^3}{3} - Ts\right) \frac{dT}{2\pi i}, \quad s \in \mathbb{R} \quad (7)$$

where \mathcal{C} is the usual contour for Airy function, running from $e^{-i\pi/3}\infty$ to $e^{i\pi/3}\infty$ on the right half of the complex plane (see Appendix A for a brief review of the Airy function).

REMARK 1. When $d = 3$, the kernel (5) with $u' = v'$, i.e. $\Pi_0(u_1, u'; v_1, u')$, coincides modulo the factor of $\sqrt{\lambda}$ with the Airy kernel $K(x, y)$ of [TW] (see (4.5) of that article). The “allowed region” of this article is analogous to the ‘bulk’ in random matrix theory, and the “caustic” of this article is analogous to the “edge of the spectrum”.

To our knowledge, this is the first result on Airy scaling asymptotics of Schrödinger eigenfunctions in dimensions $d > 1$. Theorem 1.1 is proved in Section 3. The on-diagonal result (6) is proved first in Proposition 3.3 because it is the important case for the applications to nodal sets. It is not obvious that (5) reduces to (6) when $u = v$, but this is proved by combining (7) with Lemma A.2 on products of Airy functions. The case of general E is obtained by a simple rescaling as in Section 1.5.

The isotropic Harmonic Oscillator is special even among Harmonic Oscillators because of the maximally high multiplicity of eigenvalues, and there is no direct generalization of Theorem 1.1 to eigenspace projections of other Schrödinger operators. However, we expect that the scaling asymptotics generalize if we replace eigenspaces by spectral projections for small intervals in the spectrum (work in progress). To explain the unique features of (1), we recall (Section 2.1) that $V_{\hbar N, E}$ is spanned by Hermite functions of degree N in d variables and

$$\dim V_{\hbar N, E} = \frac{1}{(d-1)!} N^{d-1} (1 + O(N^{-1})). \quad (8)$$

The high multiplicities are due to the $U(d)$ -invariance of the isotropic Harmonic Oscillator, and the periodicity of the classical Hamiltonian flow. As a result, the quantum propagator $e^{-it\hat{H}_{\hbar}/\hbar}$ is (essentially) periodic, and the Mehler formula (20) expresses the propagator as an integral over the circle. This expression is used to obtain the scaling asymptotics of $\Pi_{\hbar, E}(x, y)$ for x, y near \mathcal{C}_E . For general Harmonic Oscillators with incommensurate frequencies the eigenvalues have multiplicity one and the eigenspace projections are of a very different type. It is for this reason that we only consider the isotropic Harmonic Oscillator in this article.

Besides the edge asymptotics of [TW], the asymptotic behavior of $\Pi_{\hbar, E}$ in a $\hbar^{2/3}$ -neighborhood of the caustic is reminiscent of the scaling asymptotics of the Szegő projector in [BSZ] and of spectral projections on Riemannian manifolds [CH], which both have universal scaling limits. However, the presence of allowed and forbidden regions is a new feature of Schrödinger operators that does not occur for Laplacians or in the complex setting. If one rescales $\Pi_{\hbar, E}$ in an \hbar -neighborhood of a point in the allowed region, one would obtain results analogous to those for Laplacians on Riemannian manifolds in [CH]. But the $\hbar^{2/3}$ -scaling

asymptotics along the caustic are of a fundamentally different nature. Although there are several studies of Airy asymptotics of Wigner functions in dimension one around the caustic in phase space (originating in [Be]), we are not aware of any prior studies of the scaling asymptotics of the spectral projections kernels along the caustic in configuration space. It is an important aspect of Schrödinger equations that deserves to be studied in generality. In [HZZ3] we study the Wigner distribution $W_{h,E}$ of (3) and its Airy scaling asymptotics around the phase space energy surface $\Sigma_E = \{\frac{1}{2}(|\xi|^2 + |x|^2) = E\}$, which gives a higher dimensional generalization of [Be].

1.1. Random Hermite eigenfunctions. Theorem 1.1 has several applications to random Hermite eigenfunctions, which have recently been studied in [HZZ, PRT, IRT]. A random Hermite eigenfunction of eigenvalue $E = \hbar(N + \frac{d}{2})$ is defined by

$$\Phi_{h,E}(x) := \sum_{|\beta|=N} a_\beta \phi_{\beta,h}(x). \quad (9)$$

where $\beta = (\beta_1, \dots, \beta_d)$ and $|\beta| = \beta_1 + \dots + \beta_d$. Here the coefficients $a_\beta \sim N(0, 1)_\mathbb{R}$ are i.i.d. normal random variables and $\{\phi_{\beta,h}\}_{|\beta|=N}$ is an orthonormal basis of $V_{h,N,E}$ consisting of multivariable Hermite functions (see Section 2.1). Equivalently, we use the basis to identify $V_{h,N,E} \equiv \mathbb{R}^{\dim(V_{h,N,E})}$ and then endow $\mathbb{R}^{\dim(V_{h,N,E})}$ with the standard Gaussian measure. The Schwartz kernel of the eigenspace projection (3) is the covariance (or two-point) function of $\Phi_{h,E}$:

$$\Pi_{h,E}(x, y) := \mathbf{E}(\Phi_{h,E}(x)\Phi_{h,E}(y)) = \sum_{|\beta|=N} \phi_{\beta,h}(x)\phi_{\beta,h}(y). \quad (10)$$

The random Hermite functions $\Phi_{h,E}$ are a centered Gaussian field. Their properties are therefore completely determined by $\Pi_{h,E}$.

As a first application of Theorem 1.1, we determine the expected L^2 mass of L^2 -normalized random Hermite eigenfunctions $\Psi_{h,E} := \Phi_{h,E}(x)/\|\Phi_{h,E}\|_{L^2}$ in a metric tube $T_\delta(\mathcal{C}_E) = \{y : d(y, \mathcal{C}_E) < \delta\}$ of radius δ around the caustic. Here $d(y, \mathcal{C}_E)$ is the distance $\inf_{x \in \mathcal{C}_E} |y - x|$ from y to \mathcal{C}_E . We define the L^2 -mass-squared of an L^2 -normalized eigenfunction Ψ in the δ -tube by

$$M_2(\Psi_{h,E}, T_\delta(\mathcal{C}_E)) := \int_{T_\delta(\mathcal{C}_E)} |\Psi_{h,E}(x)|^2 dx.$$

COROLLARY 1.2. *Let $\delta = \kappa \hbar^{2/3}$, $E = 1/2$. Then the expected L^2 mass-squared of an L^2 -normalized Hermite eigenfunction in $T_\delta(\mathcal{C}_E)$ are given by*

$$\mathbf{E}M_2(\Psi_{h,E}, T_\delta(\mathcal{C}_E)) = C(d) \hbar^{d/3} \int_{-2\kappa}^{2\kappa} \int_0^\infty \text{Ai}(s + \rho) \rho^{d/2-1} d\rho ds (1 + O(\hbar^{1/3}))$$

where $C(d) = \frac{\Gamma(d)}{\Gamma(d/2)^2}$.

The L^2 mass density in the $\hbar^{2/3}$ -tube is $\hbar^{(d-2)/3}$; integration over the thin tube introduces the additional volume factor $\hbar^{2/3}$. The proof is given in §3.5.

L^p norms of Hermite eigenfunctions are studied in [Th, KT] (see also their references) and pointwise bounds on (non-dilated) Hermite functions are given near the caustic in [Sz] in dimension 1 (see Lemma 1.5.1 of [Th]). In dimension 1, the maximum of the k th (unscaled) Hermite function is achieved at a point $k^{-1/6}$ close to the caustic (turning points). This motivates the question of how L^p mass of eigenfunctions builds up around the caustic for

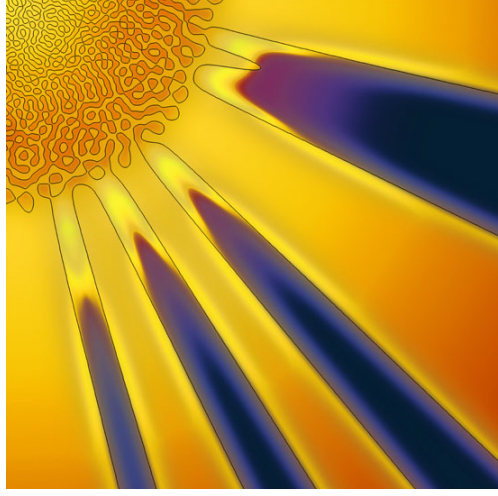


FIGURE 1. Nodal line (colored black) of $\Phi_{h,E}$, $d = 2$. (Taken from [BH])

general p in all dimensions. The same question for $p = 1$ arises in the study of nodal volumes. Eigenfunctions concentrated on a single trajectory are extremals for low L^p norms. The above Corollary shows that the mass density is constant when $d = 2$ and decays for $d \geq 3$ for a random Hermite function.

REMARK 2. For $d = 1$ it is shown in [Th], Lemma 1.5.2 that the sup norm of L^2 normalized Hermite functions H_N is $\simeq N^{-\frac{1}{12}}$. As reviewed in §2.1, the sup norm of the semi-classically scaled eigenfunctions $\phi_{N,\hbar}$ of this article (18) equal $\hbar^{-d/4}$ times the sup norm of the unscaled Hermite functions H_N . If we fix the energy level E and set $\hbar(N + \frac{1}{2}) = E$ as $\hbar \rightarrow 0, N \rightarrow \infty$, then $\|\phi_{N,\hbar}\|_{L^\infty} \simeq \hbar^{-1/4} \hbar^{1/12} E^{-1/12} \simeq \hbar^{-1/6}$, which agrees with the mass density formula above.

1.2. Applications to nodal sets of random Hermite eigenfunctions. One of our principal motivations to study the scaling asymptotics of the eigenspace projections is to understand the transition between the behavior of nodal sets of random eigenfunctions $\Phi_{h,E}$ in the allowed and forbidden regions. In particular, we study in Theorem 1.4 the average density of the nodal set

$$Z_{\Phi_{h,E}} := \{\Phi_{h,E}(x) = 0\}$$

near \mathcal{C}_E . Let us denote by $|Z_{\Phi_{h,E}}|$ the random measure of integration over $Z_{\Phi_{h,E}}$ with respect to the Euclidean hypersurface measure (the co-dimension 1 Hausdorff measure \mathcal{H}^{d-1}) of the nodal set. Thus for any measurable set $B \subset \mathbb{R}^d$,

$$|Z_{\Phi_{h,E}}|(B) := \mathcal{H}^{d-1}(B \cap Z_{\Phi_{h,E}}).$$

Its expectation $\mathbf{E}|Z_{\Phi_{h,E}}|$ is the average density or distribution of zeros and is the measure on \mathbb{R}^d defined by

$$\mathbf{E}|Z_{\Phi_{h,E}}|(B) = \int_{V_{h,E}} \mathcal{H}^{d-1}(B \cap Z_{\Phi_{h,E}}) d\gamma_{h,E},$$

where $\gamma_{h,E}$ is the Gaussian from which $\Phi_{h,E}$ is sampled. In recent work [HZZ], the authors showed that $\mathbf{E}|Z_{\Phi_{h,E}}|$ has a different order as $\hbar \rightarrow 0$ in the allowed and forbidden regions.

THEOREM 1.3 ([HZZ] 2013). *Fix $E > 0$. The measure $\mathbf{E}|Z_{\Phi_{h,E}}|$ has a density $F_{h,E}$ with respect to Lebesgue measure given by*

$$F_{h,E}(x) = \hbar^{-1} \cdot c_d \sqrt{2E - |x|^2} (1 + O(\hbar)), \quad \text{if } x \in \mathcal{A}_E \setminus \{0\} \quad (11)$$

$$F_{h,E}(x) = \hbar^{-1/2} \cdot C_d \frac{E^{1/2}}{|x|^{1/2} (|x|^2 - 2E)^{1/4}} (1 + O(\hbar)), \quad \text{if } x \in \mathcal{F}_E, \quad (12)$$

where the implied constants in the ‘O’ symbols are uniform on compact subsets of the interiors of $\mathcal{A}_E \setminus \{0\}$ and \mathcal{F}_E , and where $c_d = \frac{\Gamma(\frac{d+1}{2})}{\sqrt{d\pi}\Gamma(\frac{d}{2})}$ and $C_d = \frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(\frac{d}{2})}$ depend only on d .

Our main result on nodal sets (Theorem 1.4) gives scaling asymptotics for the average nodal density that ‘interpolate’ between (11) and (12). The computer graphics of Bies-Heller [BH] (reprinted as Figure 1 in [HZZ]) show that the nodal set in \mathcal{A}_E near the caustic $\partial\mathcal{A}_E$ consists of a large number of highly curved nodal components apparently touching the caustic while the nodal set in \mathcal{F}_E near $\partial\mathcal{A}_E$ consists of fewer and less curved nodal components all of which touch the caustic. The scaling limit of the density of zeros in a shrinking neighborhood of the caustic, or in annular subdomains of \mathcal{A}_E and \mathcal{F}_E at shrinking distances from the caustic are given Theorems 1.4 and 1.5. The varying density of zeros in $\mathcal{A}_E, \mathcal{F}_E$ near the caustic proved there is the new phenomenon related to nodal sets at issue in this article. For the expected density of zeros in the $\hbar^{2/3}$ -neighborhood of the caustic, we apply the Kac-Rice formula to Theorem 1.1; for the expected density of zeros in the \hbar^α -neighborhood of the caustic, where $0 < \alpha < 2/3$, we invoke the Kac-Rice formula together with a non-standard stationary phase method (see Propositions 6.1 and 6.5).

The nodal set of $\Phi_{h,E}$ near the caustic consists of a mixture of components from the forbidden nodal set and from the allowed nodal set (see Figure 1). To be more precise, if $\psi \in V_{h,E}$ is non-zero, then there do not exist nodal domains contained entirely in \mathcal{F}_E , where the potential $V := |x|^2/2$ is greater than the energy E , because $\Delta\psi = (V - E)\psi$ forces ψ and $\Delta\psi$ to have the same sign in \mathcal{F}_E . In a nodal domain \mathcal{D} we may assume $\psi > 0$, but then ψ is a positive subharmonic function in \mathcal{D} and cannot be zero on $\partial\mathcal{D}$ without vanishing identically. Hence, every nodal component which intersects \mathcal{F}_E must also intersect \mathcal{A}_E and therefore \mathcal{C}_E . As indicated in the computer graphics of [BH], some of these components remain in a very small tube around \mathcal{C}_E and some stretch far out into \mathcal{F}_E and most of these (apparently) stretch out to infinity. The density results of Theorem 1.3 suggest that there should only exist on average order of $\hbar^{-1/2}$ of the latter, not enough to explain the $\hbar^{-2/3}$ size of $\mathbf{E}|Z_{\Phi_{h,E}}|$ around the caustic proved in Theorem 1.4 below (see also Remark 4). Hence, one expects the main contribution to the nodal density near \mathcal{C}_E to come from nodal components living mainly in \mathcal{A}_E which cross \mathcal{C}_E .

To state our first result precisely, we fix $x \in \mathcal{C}_E$, where $E = 1/2$, and study the rescaled ensemble

$$\Phi_{h,E}^{x,\alpha}(u) := \Phi_{h,E}(x + \hbar^\alpha u)$$

and the associated hypersurface measure

$$|Z_{h,E}^{x,\alpha}|(B) = \mathcal{H}^{d-1}(\{\Phi_{h,E}^{x,\alpha}(v) = 0\} \cap B), \quad B \subset \mathbb{R}^d.$$

Our main result gives the asymptotics of $\mathbf{E}|Z_{h,E}^{x,\alpha}|$ when $\alpha = 2/3$ is in terms of the weighted Airy functions Ai_k (see (7)).

THEOREM 1.4 (Nodal set in a shrinking ball around a caustic point). *Fix $E = 1/2$ and $x \in \mathcal{C}_E$, i.e. $|x| = 1$. For any bounded measurable $B \subseteq \mathbb{R}^d$,*

$$\mathbf{E} \left| Z_{\hbar, E}^{x, 2/3} \right| (B) = \int_B \mathcal{F}(u) du,$$

where

$$\mathcal{F}(u) = (2\pi)^{-\frac{d+1}{2}} \int_{\mathbb{R}^d} |\Omega(u)^{1/2} \xi| e^{-|\xi|^2/2} d\xi (1 + O(\hbar^{1/3})) \quad (13)$$

and $\Omega = (\Omega_{ij})_{1 \leq i, j \leq n}$ is the symmetric matrix

$$\Omega_{ij}(u) = x_i x_j \left(\frac{\text{Ai}_{2-d/2}(s)}{\text{Ai}_{-d/2}(s)} - \frac{\text{Ai}_{1-d/2}^2(s)}{\text{Ai}_{-d/2}^2(s)} \right) + \frac{\delta_{ij}}{2} \frac{\text{Ai}_{-1-d/2}(s)}{\text{Ai}_{-d/2}(s)}. \quad (14)$$

where $s = 2\langle u, x \rangle$. The implied constant in the error estimate from (13) is uniform when u varies in compact subsets of \mathbb{R}^d .

REMARK 3. The leading term in \mathcal{F} is \hbar -independent and positive everywhere since the matrix $\Omega_{ij}(u)$ as a linear operator has nontrivial range. Indeed, as shown in Proposition 4.1, $\text{Ai}_{-d/2}(s) > 0$ for all integers $d \geq 2$, hence δ_{ij} term in (14) is nonzero. The matrix $(x_i x_j)_{i, j}$ in (14) is a rank 1 projection onto the x -direction. Hence, since the dimension $d \geq 2$, it cannot cancel out the second term.

REMARK 4. Theorem 1.4 says that if $x \in \mathcal{C}_E$ and $\tilde{B}_\hbar = x + \hbar^{2/3} B$ for some bounded measurable B , then

$$\mathbf{E} |Z_{\Phi_{\hbar, E}}|(\tilde{B}_\hbar) = \hbar^{2/3(d-1)} \mathbf{E} |Z_{\hbar, E}^{x, \alpha}|(B) = \hbar^{-2/3} \int_{\tilde{B}_\hbar} \mathcal{F}(\hbar^{-2/3}(y - x)) dy,$$

which shows that the average (unscaled) density of zeros in a $\hbar^{2/3}$ -tube around \mathcal{C}_E grows like $\hbar^{-2/3}$ as $\hbar \rightarrow 0$.

The choice of radius $\delta = \hbar^{\frac{2}{3}}$ is dictated by the scaling asymptotics of the associated covariance (2-point) function (10), which are stated in Theorem 1.1 and proved in §3. The rescaled random Hermite functions $\Phi_{\hbar, E}^{x, 2/3}(v)$ converge to an infinite dimensional Gaussian ensemble of solutions of a scaled eigenvalued problem, which is identified in Section 3.6.

REMARK 5. As mentioned above, the scaling asymptotics of zeros around the caustic, especially in the radial (normal) direction, is analogous to the scaling asymptotics of eigenvalues of random Hermitian matrices around the edge of the spectrum. But the scaled radial distribution of zeros of random Hermite eigenfunctions does not seem to be a determinantal process, while eigenvalues of random Hermitian matrices is determinantal.

1.3. Sub-critical shrinking of balls around caustics points. So far, we have considered in detail the rescaling of the eigenspace projections and nodal sets in a $\hbar^{2/3}$ -tube around the caustic. In [HZZ] we have studied the behavior within the allowed and forbidden regions at fixed (\hbar -independent) distance from the caustic. We refer to such regions as the ‘bulk’. In section 6, we fill in the gaps between the caustic tube and the ‘bulk’ in the allowed and forbidden regions. That is, we consider shrinking annuli around the caustic which lie outside the $\hbar^{2/3}$ -tube in a sequence of rings around the caustic. In this way, we obtain scaling results that interpolate between the bulk results of [HZZ] and the caustic scaling results above. This

is the purpose of studying sub-critical rescaling exponents, i.e. the nodal set around $x + \hbar^\alpha v$ where $x \in \mathcal{C}_E$ and $0 < \alpha < 2/3$.

THEOREM 1.5. *Fix $E = 1/2$ and $x \in \mathcal{C}_E$, and $\alpha < 2/3$. Then the rescaled expected distribution of zeros $\mathbf{E}|Z_{\hbar,E}^{x,\alpha}|(u)$ has a density $F_{\hbar,E}^{x,\alpha}(u)$ with respect to Lebesgue measure given by*

$$F_{\hbar,E}^{x,\alpha}(u) = \hbar^{-(1-\frac{3}{2}\alpha)} c_d \cdot s^{1/2} \left(1 + O(\hbar^{\alpha/2}) + O(\hbar^{1-(3/2)\alpha})\right), \text{ if } x + \hbar^\alpha u \in \mathcal{A}_E \setminus \{0\} \quad (15)$$

$$F_{\hbar,E}^{x,\alpha}(u) = \hbar^{-\frac{1}{2}(1-\frac{3}{2}\alpha)} C_d \cdot s^{-1/4} \left(1 + O(\hbar^{\alpha/2}) + O(\hbar^{1-(3/2)\alpha})\right), \text{ if } x + \hbar^\alpha u \in \mathcal{F}_E, \quad (16)$$

where $s = 2\langle u, x \rangle$, the implied constants in the 'O' symbols are uniform for u in a compact subset of \mathbb{R}^d , and c_d, C_d are positive dimensional constants.

REMARK 6. *As in Remark 4, the rescaled density $F_{\hbar,E}^{x,\alpha}$ has an extra \hbar^α factor compared with the unscaled density $\mathcal{F}_{\hbar,E}(x + \hbar^\alpha u)$ from Theorem 1.3. Hence, writing $d(y, \mathcal{C}_E)$ for the distance from y to the caustic, equations (15) and (16) agree with the results of Theorem 1.3*

$$\begin{cases} F_{\hbar,E}(x + \hbar^\alpha u) \sim \hbar^{-1} d(x + \hbar^\alpha u, \mathcal{C}_E)^{1/2} \sim \hbar^{-1+\alpha/2}, & x \in \mathcal{A}_E \\ F_{\hbar,E}(x + \hbar^\alpha u) \sim \hbar^{-1/2} d(x + \hbar^\alpha u, \mathcal{C}_E)^{-1/4} \sim \hbar^{-1/2-\alpha/4}, & x \in \mathcal{F}_E \end{cases}$$

since $-1 + \alpha/2 + \alpha = -(1 - 3\alpha/2)$, $-1/2 - \alpha/4 + \alpha = -\frac{1}{2}(1 - 3\alpha/2)$.

REMARK 7. *The two error terms $O(\hbar^{\alpha/2}), O(\hbar^{1-(3/2)\alpha})$ come from the non-standard stationary phase expansion used to prove Theorem 1.5: the $O(\hbar^{\alpha/2})$ comes from approximating the critical point of the phase function, while the $O(\hbar^{1-(3/2)\alpha})$ marks the failure of the stationary phase expansion as α approaches $2/3$.*

The main ingredient in the proof is the sub-critical scaling asymptotics of the covariance function (10) in Propositions 6.1 resp. 6.5 around points in \mathcal{A}_E , resp. \mathcal{F}_E .

1.4. Nodal set intersections with the caustic. Our next result measures the density of intersections of the nodal set with the caustic. This is much simpler than measuring the density in shrinking tubes or annuli, since it is not necessary to rescale the covariance kernel (10). For an open set $B \subset \mathcal{C}_E$ we consider

$$\mathcal{H}^{d-2}(B \cap Z_{\Phi_{\hbar,E}}).$$

When $d = 2$ this means to count the number of nodal intersections with the caustic. Since the Gaussian measure is $SO(d)$ invariant and the caustic is a sphere, the average nodal density along the caustic is constant.

THEOREM 1.6. *Fix $d \geq 2$, $E = 1/2$, and define the constant*

$$\mathcal{F}_{\mathcal{C}_E,d} = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{2\pi}\Gamma\left(\frac{d-1}{2}\right)} \cdot \left(\frac{\text{Ai}_{-1-d/2}(0)}{\text{Ai}_{-d/2}(0)}\right)^{1/2},$$

where Ai_k are defined in (7). Then, as $\hbar \rightarrow 0$, for any open set $B \subset \mathcal{C}_E$

$$\mathbf{E}\mathcal{H}^{d-2}(B \cap Z_{\Phi_{\hbar,E}}) = \hbar^{-\frac{2}{3}} \mathcal{F}_{\mathcal{C}_E,d} \text{Vol}_{S^{d-1}}(B) (1 + O(\hbar^{1/3})).$$

In particular, if $d = 2$, $\mathbf{E}(\#Z_{\Phi_{\hbar,E}} \cap \mathcal{C}_E) = C_0 \hbar^{-2/3} (1 + O(\hbar^{1/3}))$ where

$$C_0 = \sqrt{2} (\text{Ai}_{-2}(0)/(\text{Ai}_{-1}(0)))^{1/2} > 0.$$

The proof is to use a Kac-Rice formula for the expected number of intersections of the nodal set with the caustic. The relevant covariance kernel is the restriction of Ω_{ij} in (14) to the tangent plane of the caustic (hence the radial component $x_i x_j$ drops out). It is analogous to the formula in [TW] (Proposition 3.2) for the expected number of intersections of nodal lines with the boundary of a plane domain, and we therefore omit its proof.

1.5. Outline of the proofs. As mentioned above, the proofs of Theorems 1.4, 1.5, and 1.6 are based on a detailed analysis of the Kac-Rice formula (Lemma 2.2 in §2.4), which gives a formula for the average density of zeros at x in terms of a covariance matrix Ω_x depending only on $\Pi_{h,E}(x, y)$ and its derivatives

$$d_x \Pi_{h,E}(x, y), d_y \Pi_{h,E}(x, y), d_x d_y \Pi_{h,E}(x, y)$$

restricted to the diagonal. The kernels behave differently depending on the position of x relative to the caustic and have contour integral representations of the following type

$$\int_{\mathcal{C}} \frac{1}{z^k} \exp\left(\frac{-uz}{\hbar^{1-\alpha}} + \frac{z^3}{\hbar}\right) dz$$

where \mathcal{C} is the usual Airy contour (Appendix A), $u \in \mathbb{C}$ is a parameter, and we simplified the phase function and the amplitude by keeping only the leading term. See (27) for the precise formulas. The critical points of the simplified phase function are $z_{\pm} = \pm \sqrt{u/3} \hbar^{\alpha/2}$. If $\alpha = 0$, we would have two separate critical points, which were analyzed in our previous paper [HZZ]. If $0 < \alpha < 2/3$, then as $\hbar \rightarrow 0$ the two critical points z_{\pm} starts to move closer, with $|z_+ - z_-| \sim \hbar^{\alpha/2}$, and each corresponds with a Gaussian bump of width $\hbar^{1/2-\alpha/4}$. Hence the two bumps will start overlapping when $\alpha = 2/3$. In more details, the integral near each critical point can be roughly evaluated as follows

$$\begin{aligned} C_h^{-1} \int (z_c + \eta)^{-k} e^{-\frac{\eta^2}{\hbar^{1-\alpha/2}}} d\eta &= C_h^{-1} \int ((z_c)^{-k} + C(k)(z_c)^{-k-2}\eta^2 + \dots) e^{-\frac{\eta^2}{\hbar^{1-\alpha/2}}} d\eta \\ &= (z_c)^{-k} (1 + C(k)(z_c)^{-2}\hbar^{1-\alpha/2} + \dots) = (z_c)^{-k} (1 + O(\hbar^{1-(3/2)\alpha})) \end{aligned}$$

where $C_h = \int e^{-\frac{\eta^2}{\hbar^{1-\alpha/2}}} d\eta$ is a normalization constant. The point of the above sketch computation is to show that due to the singularity of z^{-k} , the error term is enhanced to $O(\hbar^{1-(3/2)\alpha})$ from the naive expectation $O(\hbar/Hess(\Phi)) = O(\hbar^{1-\alpha/2})$. Hence the break down of the expansion at $\alpha = 2/3$. Another way to see the break down of stationary phase at $\alpha = 2/3$, which we do not go into detail in the paper¹, is to rescale z centered at $z = 0$ rather than at $z = z_c$. Namely if we set $z = \hbar^{\alpha/2}\eta$, then we get

$$\int \eta^{-k} \exp\left(\frac{-u\eta + \eta^3}{\hbar^{1-(3/2)\alpha}}\right) d\eta$$

Hence we see at $\alpha = 2/3$, the \hbar factor in the exponent is unity, hence we do not have a small parameter to do asymptotic expansion, and have to express the result in terms of the weighted Airy functions Ai_{-k} defined in (7).

¹The reason we did not adopt the rescaling $z = \hbar^{\alpha/2}\eta$ is that, in the actual calculation, we have higher order terms z^4, z^5 etc in the phase function, and after the change of variable we get $e^{\hbar^{-1}z^4} = e^{\hbar^{-1+2\alpha}\eta^4}$, which cannot be treated perturbatively unless $\alpha > 1/2$. To use this method, one needs to use the Malgrange preparation theorem (see [FW]) to get rid of the higher order terms, which we choose not to do in this paper.

Notation. To get rid of $2E$ factors, we will set $2E = 1$ for the rest of the article, and drop the E subscript. The general case can be obtained by the following replacement

$$x \mapsto x' := x/\sqrt{2E}, \quad p \mapsto p' := p/\sqrt{2E}, \quad \hbar \mapsto \hbar' := \hbar/2E \quad (17)$$

and

$$\Pi_{h,E}(x, y) = (2E)^{-d/2} \Pi_{h'}(x', y'), \quad \Omega_{h,E} = (2E)^{-1} \Omega_{h'}(x'), \quad F_{h,E}(x) = (2E)^{-1/2} F_{h'}(x').$$

For general E , the dilation factor $\delta = C\hbar^\alpha$ in Theorem 1.5 should be changed to $C(\hbar/2E)^\alpha$. We will also abbreviate throughout $V_{h_N,E} = V_{h,E}$ with the understanding that E is fixed so that $\hbar = \hbar_N = E/N - d/2$.

2. BACKGROUND

We follow the notation of [HZZ] and refer there for background on the isotropic Harmonic Oscillator, its spectrum and its spectral projections. We also refer to that article for background on the Kac-Rice formula and other fundamental notions on Gaussian random Hermite eigenfunctions. In this section, we recall some of the basic definitions and facts that are used in the proofs of the main results.

2.1. Eigenspaces. Fix $E = 1/2$. An orthonormal basis of the eigenspace $V_{h_N,E}$ (2) is given by

$$\phi_{\beta,h}(x) = \hbar^{-d/4} p_\beta(\hbar^{-1/2} x) e^{-x^2/2\hbar}, \quad (18)$$

where $\beta = (\beta_1, \dots, \beta_d) \geq (0, \dots, 0)$ is a d -dimensional multi-index with $|\beta| = \sum_i \beta_i = N$ and $p_\beta(x) = \prod_{j=1}^d p_{\beta_j}(x_j)$ is the product of the Hermite polynomials p_k (of degree k) in one variable, with the normalization that $\int_{\mathbb{R}} p_k(x)^2 e^{-x^2} dx = 1$. The eigenvalue of $\phi_{\beta,h}$ is given by

$$H_h \phi_{\beta,h} = \hbar(|\beta| + d/2) \phi_{\beta,h}. \quad (19)$$

The multiplicity of the eigenvalue $\hbar(|\beta| + d/2)$ is the partition function of $|\beta|$, i.e. the number of $\beta = (\beta_1, \dots, \beta_d) \geq (0, \dots, 0)$ with a fixed value of $|\beta|$. Hence

$$\dim V(\hbar, E) = \binom{N + d - 1}{d - 1} = \frac{1}{(d - 1)!} N^{d-1} (1 + O(N^{-1})).$$

For further background and notation we refer to [HZZ].

2.2. Mehler Formula for the propagator. The Mehler formula gives an explicit expression for the Schwartz (Mehler) kernel

$$U_h(t, x, y) = e^{-\frac{i}{\hbar} t \hat{H}_h}(x, y) = \sum_{N=0}^{\infty} e^{-\frac{i}{\hbar} t E_N} \Pi_{h,E_N}(x, y)$$

of the propagator, $e^{-\frac{i}{\hbar} t \hat{H}_h}$. The Mehler formula [F] reads

$$U_h(t, x, y) = e^{-\frac{i}{\hbar} t \hat{H}_h}(x, y) = \frac{1}{(2\pi i \hbar \sin t)^{d/2}} \exp \left(\frac{i}{\hbar} \left(\frac{|x|^2 + |y|^2 \cos t}{2 \sin t} - \frac{x \cdot y}{\sin t} \right) \right), \quad (20)$$

where $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$. The right hand side is singular at $t = 0$. It is well-defined as a distribution, however, with t understood as $t - i0$. Indeed, since \hat{H}_h has a positive spectrum

the propagator U_h is holomorphic in the lower half-plane and $U_h(t, x, y)$ is the boundary value of a holomorphic function in $\{\Im t < 0\}$.

2.3. Spectral projections. We also use that the spectrum of \widehat{H}_h is easily related to the integers $|\beta|$. The operator with the same eigenfunctions as \widehat{H}_h and eigenvalues $\hbar|\beta|$ is often called the number operator, $\hbar\mathcal{N}$. If we replace $U_h(t)$ by $e^{-\frac{it}{\hbar}\mathcal{N}}$ then the spectral projections $\Pi_{h,E}$ are simply the Fourier coefficients of $e^{-\frac{it}{\hbar}\mathcal{N}}$. In [HZZ], we used the related formula,

$$\Pi_{h,E}(x, y) = \int_{-\pi-i\epsilon}^{\pi-i\epsilon} U_h(t, x, y) e^{\frac{it}{\hbar}E} \frac{dt}{2\pi}, \quad (21)$$

where, as before, $E = \hbar(N + d/2)$. The integral is independent of ϵ . Using the Mehler formula (20) we obtain a rather explicit integral representation of (10). If we introduce a new complex variable $z = e^{-it}$, then the above integral can be written as

$$\Pi_{h,E_N}(x, y) = \oint_{C_\epsilon} \left(\frac{z}{\pi\hbar(1-z^2)} \right)^{d/2} e^{-\frac{1}{\hbar} \left(\frac{1+z^2}{1-z^2} \frac{x^2+y^2}{2} - \frac{2z}{1-z^2} x \cdot y + E \log z \right)} \frac{dz}{2\pi i z} \quad (22)$$

where the contour is a circle $C_\epsilon = \{|z| = 1 - \epsilon\}$ traversed counter-clockwise.

2.4. The Kac-Rice Formula. As with Theorem 1.3, the proofs of Theorems 1.4, 1.5 and 1.6 are based on the Kac-Rice formula [AW, Thm. 6.2, Prop. 6.5] for the average density of zeros. The Kac-Rice formula is the formula for the pushforward of the Gaussian measure on the random Hermite functions under the evaluation maps $ev_x(\phi) = (\phi(x), \nabla\phi(x))$. It is valid at $x \in \mathbb{R}^d$ as long as the so-called 1-jet spanning property holds at x . Namely, $ev_x : V_{h_N,E} \rightarrow \mathbb{R}^{d+1}$ is surjective, or equivalently, the $(d+1) \times (d+1)$ covariance matrix $\Sigma_x = \text{Cov}(\Phi_{h,E}(x), \nabla_x \Phi_{h,E}(x))$ of values and gradients is invertible. We now verify that the condition for the validity of the Kac-Rice formula holds.

PROPOSITION 2.1. *For any $x \in \mathbb{R}^d \setminus \{0\}$, the 1-jet evaluation map*

$$ev_x : V_{h_N,E} \rightarrow \mathbb{R}^{d+1}, \quad \varphi \mapsto (\varphi(x), \partial_1 \varphi(x), \dots, \partial_d \varphi(x))$$

is surjective, where $V_{h_N,E}$ is defined in Eq (2). Equivalently, if $V_{h_N,E}$ is equipped with a standard Gaussian measure γ_V induced by the inner product on $V_{h_N,E}$, then its pushforward under ev_{x} is a non-degenerate Gaussian measure on \mathbb{R}^{d+1} .*

Proof. Fix an orthonormal basis $\{\phi_j\}_{j=1}^{d_N}$ of $V_{h_N,E}$, then ev_x can be written as a $(d+1) \times d_N$ matrix M_x , where $d_N = \dim V_{h_N,E}$. Showing ev_x is surjective is equivalent to showing M_x has rank $d+1$, or $\Sigma_x := M_x M_x^t$ is a non-degenerate $(d+1) \times (d+1)$ square matrix. By definition, Σ_x is the covariance matrix of the Gaussian measure $ev_{x*}(\gamma_V)$. Hence, the two statements in the proposition are equivalent.

Recall that $\Phi_{h,E}$ is the Gaussian random variable valued in $V_{h_N,E}$ with measure γ_V . We then express Σ_x in polar coordinates $x = (r, \omega)$ where $\omega \in S^{d-1}$ and $r \in \mathbb{R}_{>0}$. The first observation is that Σ_x is block diagonal if we break it up into its radial part and angular part,

$$\Sigma_{(r,\omega)} = \left(\begin{array}{cc} \text{Cov}(\Phi_{h,E}, \partial_r \Phi_{h,E}) & 0 \\ 0 & \text{Cov}(\nabla_\omega \Phi_{h,E}, \nabla_\omega \Phi_{h,E}) \end{array} \right) \Big|_{(r,\omega)}. \quad (23)$$

Indeed, the $(\phi, \nabla_\omega \phi)$ block is $\nabla_\omega \Pi_{h,E}(r, \omega, r, \omega')|_{\omega=\omega'} = 0$ and $\Pi_{h,E}(x, x)$ is invariant under rotations in $SO(d)$. For the same reason the mixed (r, ω) derivatives are zero. The block-diagonality may be expressed in an invariant form by combining the second derivative block into the Riemannian metric

$$g_x := d_x \otimes d_y \Pi_{h_N, E}(x, y)|_{x=y} = \sum_{|\beta|=N} d\phi_{\beta, h}(x) \otimes d\phi_{\beta, h}(x) \quad (24)$$

of the process. Due to the $O(d)$ symmetry, the metric has the form $G(r) dr^2 + H(r) g_{S^{d-1}}$ where $g_{S^{d-1}}$ is the standard metric of S^{d-1} and $G(r), H(r)$ are radial functions. This is equivalent to the statement that the angular block is orthogonal to the radial block.

Next we check that the angular derivative block is invertible, i.e. that $H(r) > 0$. The isotropy group of x is $SO(d-1)$ acting in the tangent space to the sphere centered at the origin through x . By the $SO(d-1)$ symmetry

$$\text{Cov}(\nabla_\omega \Phi_{h,E}, \nabla_\omega \Phi_{h,E})|_{(r,\omega)} = H(r) \cdot I_{d-1}$$

where I_{d-1} is the diagonal matrix. Taking trace on both sides of the above equation, we get

$$(d-1)H(r) = \sum_{j=1}^{d_N} |\nabla_\omega \phi_j(r, \omega)|^2.$$

If $H(r) = 0$ for some $r = r_0$, then it means every Hermite function in $V_{h_N, E}$ is constant at the sphere with radius $r = r_0$, which is absurd since any product Hermite functions in Eq. (18) is not constant at any sphere with positive radius. To complete the proof, we need to show that the upper 2×2 block

$$\text{Cov}(\Phi_{h,E}, \partial_r \Phi_{h,E}) = \begin{pmatrix} \Pi_{h,E}(x, x) & \frac{1}{2} \partial_r \Pi_{h,E}(x, x) \\ \frac{1}{2} \partial_r \Pi_{h,E}(x, x) & \partial_r \otimes \partial_{r'} \Pi_{h,E}(x, y)|_{r=r'} \end{pmatrix}$$

is invertible. By the same argument as in the beginning of the proof, it is equivalent to showing the following linear map

$$V_{h_N, E} \rightarrow \mathbb{R}^2, \quad \phi \mapsto (\phi, \partial_r \phi)|_{(r,\omega)}$$

is surjective for any $(r, \omega) \in \mathbb{R}^d \setminus \{0\}$. We will provide two functions $\phi^{(1)}, \phi^{(2)}$ in $V_{h_N, E}$, whose span surjects to \mathbb{R}^2 . Without loss of generality, we may assume ω is in the positive x_1 direction, then $\partial_r \phi = \partial_{x_1} \phi$. If N is even, we take

$$\begin{aligned} \phi^{(1)}(r, \omega) &= \phi_{(0, N, 0, \dots, 0), h}(r, 0, \dots, 0) = c_1 e^{-r^2/2h} \\ \phi^{(2)}(r, \omega) &= \phi_{(2, N-2, 0, \dots, 0), h}(r, 0, \dots, 0) = c_2 (r^2/h - 1) e^{-r^2/2h}, \end{aligned}$$

and if N is odd, we take

$$\begin{aligned} \phi^{(1)}(r, \omega) &= \phi_{(1, N-1, 0, \dots, 0), h}(r, 0, \dots, 0) = c_1 ((h^{-1/2}r) e^{-r^2/2h}) \\ \phi^{(2)}(r, \omega) &= \phi_{(3, N-3, 0, \dots, 0), h}(r, 0, \dots, 0) = c_2 ((h^{-1/2}r)^3 - 3(h^{-1/2}r)) e^{-r^2/2h}, \end{aligned}$$

where we used that the 1D Hermite functions satisfy $\phi_{\ell, h}(0) \neq 0$ if ℓ is even, and c_1, c_2 are r -independent constant. Rescaling the variable $r \mapsto r\hbar^{1/2}$ to get rid of the \hbar dependence and

omitting the c_i factors, we may verify that the image of $\phi^{(1)}, \phi^{(2)}$ are independent:

$$\det \begin{pmatrix} \phi^{(1)} & \partial_r(\phi^{(1)}) \\ \phi^{(2)} & \partial_r(\phi^{(2)}) \end{pmatrix} = \det \begin{pmatrix} fe^{-r^2/2} & \partial_r(fe^{-r^2/2}) \\ ge^{-r^2/2} & \partial_r(ge^{-r^2/2}) \end{pmatrix} = e^{-r^2} \det \begin{pmatrix} f & \partial_r f \\ g & \partial_r g \end{pmatrix}$$

where N is even, $\{f, g\} = \{1, r^2 - 1\}$, if N is odd, $\{f, g\} = \{r, r^3 - 3r\}$. A direct computation shows that the above determinant is non-zero. Hence $\text{Cov}(\Phi_{h,E}, \partial_r \Phi_{h,E})$ is invertible, and the covariance matrix in Eq (23) is non-degenerate. This finishes the proof for the proposition. \square

We then have,

LEMMA 2.2 (Kac-Rice for Gaussian Fields). *Let $\Phi_{h,E}$ be the random Hermite eigenfunction of \hat{H}_h with eigenvalue E as in (9). Then the density of zeros of $\Phi_{h,E}$ is given by*

$$F_{h,E}(x) = (2\pi)^{-\frac{d+1}{2}} \int_{\mathbb{R}^d} |\Omega^{1/2}(x)\xi| e^{-|\xi|^2/2} d\xi, \quad (25)$$

where $\Omega(x)$ is the $d \times d$ matrix

$$\begin{aligned} \Omega_{ij}(x) &= (\partial_{x_i} \partial_{y_j} \log \Pi_{h,E})(x, x) \\ &= \frac{(\Pi_{h,E} \cdot \partial_{x_i} \partial_{y_j} \Pi_{h,E})(x, x) - (\partial_{x_i} \Pi_{h,E} \cdot \partial_{y_j} \Pi_{h,E})(x, x)}{\Pi_{h,E}(x, x)^2} \end{aligned} \quad (26)$$

and $\Pi_{h,E}(x, y)$ is the kernel of eigenspace projection (10).

We refer to [HZZ] for background. The main task in proving results on zeros near the caustic is therefore to work out the asymptotics of $\Pi_{h,E}(x, x)$ and its derivatives there.

3. THE TUBE REGION FOR $\alpha = \frac{2}{3}$: PROOFS OF THEOREMS 1.1 AND 1.4

This section is the heart of the paper, in which we determine the Airy scaling asymptotics of the eigenspace projections (10) and of the Kac-Rice matrix (26) in a $\hbar^{2/3}$ -tube around the caustic. We also find the scaling asymptotics of the derivatives of the kernel and prove the Kac-Rice formula of Theorem 1.4. We begin with an outline of the proof of Theorem 1.1 (and Theorem 1.5 since its proof follows a similar pattern).

3.1. Outline of the proof of Theorems 1.1 and 1.5. The proofs of Theorems 1.1 and 1.5 are based on a steepest descent analysis of the contour integral (22) for the eigenspace projection kernel. They involve a number of tricky technical steps, which we now sketch.

We fix $E = 1/2$ (i.e. set $\hbar = \hbar_N = (2N + d)^{-1}$) and drop the E subscript. Using (22), the spectral projector Π_{\hbar} and the derivatives appearing in the Kac-Rice matrix Ω (given in (26)) can be written as

$$\begin{aligned} \Pi_{\hbar}(x, x) &= \oint_{C_{\epsilon}} A(z) e^{\Phi(z)/\hbar} dz \\ \partial_{x_i} \Pi_{\hbar}(x, x) &= \oint_{C_{\epsilon}} \left(-\frac{x_i}{\hbar} \frac{1-z}{1+z} \right) A(z) e^{\Phi(z)/\hbar} dz \\ \partial_{x_i} \partial_{y_j} \Pi_{\hbar}(x, x) &= \oint_{C_{\epsilon}} \left[\frac{x_i x_j}{\hbar^2} \left(\frac{1-z}{1+z} \right)^2 + \frac{\delta_{ij}}{\hbar} \frac{2z}{1-z^2} \right] A(z) e^{\Phi(z)/\hbar} dz. \end{aligned} \quad (27)$$

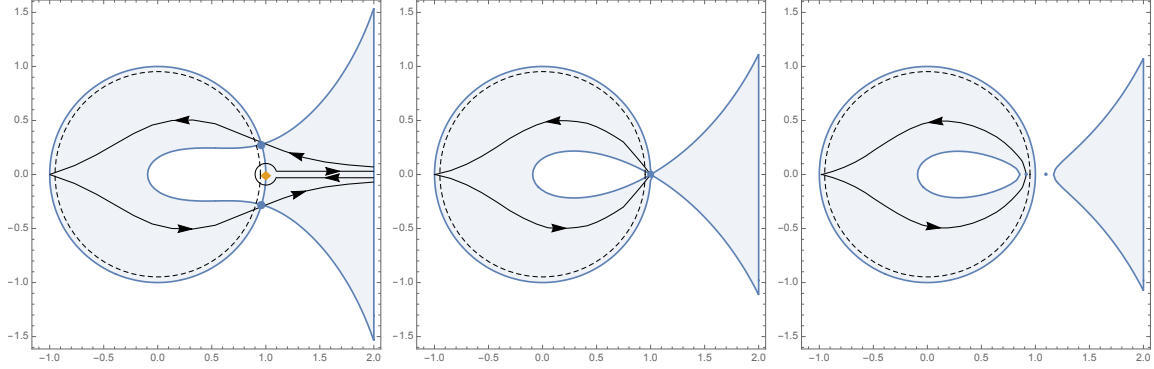


FIGURE 2. The contour deformation for $|x|^2 < 1$, $|x|^2 = 1$ and $|x|^2 > 1$. Regions where $\text{Re}(\Phi) < 0$ are shaded blue.

The phase function $\Phi(z)$ and the amplitude $A(z)$ are

$$\Phi(z) = -\frac{1-z}{1+z}|x|^2 - \frac{1}{2}\log z, \quad A(z)dz = \left(\frac{z}{\pi\hbar(1-z^2)}\right)^{d/2} \frac{dz}{2\pi iz}. \quad (28)$$

Note that the integrand $A(z)e^{\Phi(z)/\hbar}$ is defined on

$$S = \mathbb{C} \setminus ((\infty, -1] \cup [1, \infty) \cup \{0\}). \quad (29)$$

Indeed, the term $e^{-\log(z)/(2\hbar)}$ from $e^{\Phi(z)/\hbar}$ and $z^{d/2-1}$ from $A(z)$ combine to give an integer total power of z

$$z^{\frac{d}{2} - \frac{1}{2\hbar} - 1} = z^{-N-1},$$

where $N = \frac{1}{2}(\hbar^{-1} - d)$ is the degree of the Hermite functions in the $\frac{1}{2}$ -eigenspace of \widehat{H}_\hbar . Observe that the integrand has singularities at $z = \pm 1$ and the critical points

$$z_\pm = 1 \pm 2|x|\sqrt{|x|^2 - 1} + 2(|x|^2 - 1) \quad (30)$$

of $\Phi(z)$ do not lie on C_ϵ . If x is in the \hbar^α neighborhood of the caustic \mathcal{C}_E (the unit circle), say

$$|x|^2 = 1 + \hbar^\alpha s, \quad (31)$$

then

$$z_\pm = 1 \pm 2\hbar^{\alpha/2}\sqrt{s} + O(\hbar^\alpha),$$

and the Hessian of $\Phi(z)$ at the critical points is

$$\Phi''(z_\pm) = \mp 2^{-3/2}\hbar^{\alpha/2}\sqrt{s} + O(\hbar^\alpha).$$

The proofs of Theorems 1.1 and 1.5 proceed schematically in three steps:

- (i) Deform the contour such that it passes through the critical points of the phase function $\Phi(z)$, and wraps around the singular points of the amplitude function $A(z)$,
- (ii) Show that the contribution from the contour away from the critical points and singular points are irrelevant.
- (iii) Calculate the leading term contribution from the contour near the critical points and the singular points.

The contour deformations we use in step (i) are shown in Figure 2, where the panels from left to right correspond to the allowed region ($x \in \mathcal{A}_E$ and $\alpha < 2/3$), the caustic region ($\alpha = 2/3$) and the forbidden region ($x \in \mathcal{F}_E$ and $\alpha < 2/3$) respectively. The black oriented lines are the deformed contours, and the dashed line corresponds to the original contour. The blue regions are those where $\text{Re}(\Phi) < 0$ and hence the integrand is rapidly decaying in \hbar . Note that

- (1) In the allowed region, the critical points lie on the unit circle; in the caustic region $|x|^2 = 1$, the two critical point merge at $z = 1$; in the forbidden region, the two critical points are real, lie on the opposite sides of 1, and the deformed contour passes through the critical point inside the circle.
- (2) In all three cases, the contour goes to the point $z = -1$ with finite slope, and $\text{Re}\Phi(z) \rightarrow -\infty$, in agreement with the contour following the downward gradient flow of $\text{Re}\Phi(z)$.
- (3) In the allowed region case (left panel), there is an additional ‘key-hole’ contour, caused by the singularity of $A(z)$ at $z = 1$.

There are two non-standard aspects in our stationary phase integral when $|x| \approx 1$, which are particularly important for the proof of Theorem 1.5. The first is the coalescing of the critical points when $|x| = 1$, which causes the usual stationary phase method with quadratic phase function to break down. The second is the singularity in the amplitude $A(z)$ at $z = \pm 1$. The singularity at $z = -1$ is less problematic, since the phase function $\text{Re}(\Phi(z)) \rightarrow -\infty$ as $z \rightarrow -1$ along the steepest descent path, making the integral $e^{\Phi(z)/\hbar} A(z)$ convergent. The singularity of $A(z)$ at $z = 1$, however, coincides with the critical point $z_c = 1$ if $|x|^2 = 1$ right on the caustic.

A key point in the proof of Theorem 1.1 (when $|x|^2 = 1 + \hbar^{2/3}s$) is the form of the deformed contour C_ϵ in the central panel of Figure 2. Namely, in an $\hbar^{1/3}$ neighborhood of $z = 1$ the deformed contour becomes (up a sign change) to the Airy contour \mathcal{C} (see the left panel in Figure 3). The integral over this portion of the contour will turn out to give the leading order behavior of $\Pi_\hbar(x, x)$. The phase function Φ , when properly rescaled, becomes the Airy phase and it is in this way that the primitives and derivatives Ai_k of the Airy function appear in the statement of Theorem 1.1 (see Section 1.5 as well as the beginning of the proof of Proposition 3.3 for more on this point).

In the proofs of both Theorem 1.1 and 1.5 we implement step (ii) for each critical point z_c by using the Excision Lemma 3.1 to localize, to leading order, the integral (27) (with the contour deformed as above) to a neighborhood of z_c

$$\Gamma_{z_c, \delta} := \{z \in \Gamma \mid 0 > [\text{Re}(\Phi(z)) - \text{Re}(\Phi(z_c))]/\hbar > -\hbar^{-\delta}\}$$

for some $\delta > 0$.

Finally, in step (iii), to get the leading order order behavior of (27), we evaluate the localized integral over $\Gamma_{z_c, \delta}$ by Taylor expanding $A(z)$ and $\Phi(z)$ around z_c and use the Trimming Lemma 3.2 to estimate the error term.

3.2. Two Technical Lemmas. As discussed above, after deforming the contour to good position, we simplify the integrals in two ways. Since the simplifications are a bit technical, we prove the relevant Lemmas before getting into the proof of Theorem 1.1. The two Lemmas are general statements about oscillatory integrals with complex phase.

LEMMA 3.1 (Excision Lemma). *Let*

$$I_h = \int_{\Gamma_h} f_h(z) e^{\Phi_h(z)} dz$$

where for each $h \in (0, h_0)$ we have the following conditions satisfied

(1) Γ_h is a compact smooth curve in \mathbb{C} , and there are $C_1, \alpha > 0$, such that $\text{Length}(\Gamma_h) < C_1 h^{-\alpha}$

(2) $f_h(z)$ is analytic in a neighborhood of Γ_h , and there are $C_2, \beta > 0$, such that

$$\sup_{h \in (0, h_0), z \in \Gamma_h} |f_h(z)| h^\beta < C_2$$

(3) $\Phi_h(z)$ is analytic in a neighborhood of Γ_h , and there are constants $C_3, \gamma > 0$, such that

$$\sup_{h \in (0, h_0), z \in \Gamma_h} \text{Re}(\Phi_h(z)) h^\gamma < -C_3$$

Then

$$I_h = O(h^\infty)$$

Proof. For all $h \in (0, h_0)$, we have

$$|I_h| \leq \int_{\Gamma_h} |f_h(z)| e^{\text{Re}(\Phi_h(z))} dz \leq C_2 h^{-\beta} C_1 h^{-\alpha} e^{-C_3 h^{-\gamma}} = O(h^\infty)$$

□

LEMMA 3.2 (Trimming Lemma). *Fix $a, C, m > 0$ and $\epsilon \in (0, a/m)$. Let Γ be a contour in \mathbb{C} that goes to infinity along $\arg(z) = \theta_\pm$, $f(z)$ a holomorphic function in a neighborhood of Γ , and let $A_h(z), B_h(z)$ be smooth functions in a neighborhood of Γ . Assume that*

$$|f(z)| < C(1 + |z|^m), \quad |A_h(z)| + |B_h(z)| \leq C h^a (1 + |z|^m), \quad \forall z \in \Gamma \cap D(h^{-\epsilon}),$$

where $D(r)$ is the disk of radius r centered at the origin. For each $h > 0$, let

$$I_h = \int_{\Gamma \cap D(h^{-\epsilon})} (f(z) + B_h(z)) e^{\Phi(z) + A_h(z)} dz, \quad \Phi(z) = \sum_{k=0}^d a_k z^k,$$

with $\text{Re}(a_d e^{id\theta_\pm}) < 0$. Then, as $h \rightarrow 0$, we have

$$I_h = \int_{\Gamma} f(z) e^{\Phi(z)} dz + O(h^a).$$

Proof. For h small enough such that $2C h^{a-m\epsilon} < 1$, we have

$$|A_h(z)| + |B_h(z)| \leq C h^a (1 + |z|^m) \leq C(h^a + h^{a-m\epsilon}) \leq 1, \quad e^{|A_h(z)|} \leq e \quad \forall z \in \Gamma \cap D(h^{-\epsilon}).$$

Let $I_0 = \int_{\Gamma} f(z) e^{\Phi(z)} dz$. By assumption, $\text{Re}(\Phi(z)) < 0$ as $z \rightarrow \infty$ along Γ and $f(z)$ has only polynomial growth, hence I_0 is finite. Next, define

$$I_{h,s} = \int_{\Gamma \cap D(h^{-\epsilon})} (f(z) + s B_h(z)) e^{\Phi(z) + s A_h(z)} dz, \quad s \in [0, 1].$$

From Taylor's formula, we have

$$|I_{h,1} - I_{h,0}| \leq \sup_{s \in [0,1]} \left| \frac{d}{ds} I_{h,s} \right|,$$

and there exists $C' > 0$ so that

$$\begin{aligned} \left| \frac{d}{ds} I_{h,s} \right| &= \left| \int_{\Gamma \cap D(h^{-\epsilon})} [B_h(z) + (f(z) + sB_h(z))A_h(z)] e^{\Phi(z) + sA_h(z)} dz \right| \\ &\leq \int_{\Gamma \cap D(h^{-\epsilon})} C' \hbar^a (1 + |z|^{2m}) e^{|A_h(z)|} e^{\operatorname{Re}(\Phi(z))} |dz| \\ &\leq e C' \hbar^a \int_{\Gamma} (1 + |z|^{2m}) e^{\operatorname{Re}(\Phi(z))} |dz| \leq C' e \hbar^a \end{aligned}$$

Finally, we note that for some $C'' > 0$

$$|I_0 - I_{h,0}| = \left| \int_{\Gamma \setminus D(h^{-\epsilon})} f(z) e^{\Phi(z)} dz \right| = O(e^{-C'' \hbar^{-d\epsilon} + \epsilon}) = O(\hbar^\infty)$$

This shows that $|I_h - I_0| = O(\hbar^a)$ and completes the proof. \square

With these preparations, we are now ready to prove Theorem 1.1.

3.3. Diagonal scaling asymptotics of Theorem 1.1. The proof of the scaling asymptotics (4) of the eigenspace projection kernels is almost the same along the diagonal $u = v$ as it is for $u \neq v$. Since the on-diagonal result

$$\Pi_{h,1/2}(x_0 + \hbar^{2/3}u, x_0 + \hbar^{2/3}u) = \hbar^{-2d/3+1/3} \Pi_0(u, u) (1 + O(\hbar^{1/3})) \quad (32)$$

and its analogs for the derivatives $\partial_{x_i} \Pi_h(x, x)$, $\partial_{x_i} \partial_{y_j} \Pi_h(x, x)$ are the key ingredients in obtaining the scaled Kac-Rice formulae of Theorem 1.4, we state them separately in the following Proposition. We then indicate in Section 3.4 the additional steps required to obtain the off-diagonal statement of Theorem 1.1.

PROPOSITION 3.3. *Let $|x|^2 = 1 + \hbar^{2/3}s$ for any $s \in \mathbb{R}$, then*

$$\Pi_h(x, x) = 2^{-d+1} \pi^{-d/2} \hbar^{(1-2d)/3} \operatorname{Ai}_{-d/2}(s) (1 + O(\hbar^{1/3})). \quad (33)$$

Moreover, the entries of the Kac-Rice matrix (25) have the scaling asymptotics,

$$\Omega_{ij} = \hbar^{-4/3} \left\{ x_i x_j \left[\frac{\operatorname{Ai}_{2-d/2}(s)}{\operatorname{Ai}_{-d/2}(s)} - \left(\frac{\operatorname{Ai}_{1-d/2}(s)}{\operatorname{Ai}_{-d/2}(s)} \right)^2 \right] + 2^{-1} \delta_{ij} \frac{\operatorname{Ai}_{-1-d/2}(s)}{\operatorname{Ai}_{-d/2}(s)} \right\} (1 + O(\hbar^{1/3})). \quad (34)$$

The implied constants are uniform when s varies over a compact set.

Proof. First, we use the integral expression (27) to write $\Pi_h(x, x) = \int_{C_\epsilon} A(z) e^{\Phi(z)/\hbar}$, where $\Phi(z)$ and $A(z)$ are given in (28) and $C_\epsilon = \{|z| = 1 - \epsilon\}$ counter clockwise.

Then, we deform the contour C_ϵ along the steepest descent path, except bending it a bit near $z = 1$ singularity. More precisely, fix $1 \gg \delta > 0$, we then define

$$z'_\pm = 1 - \hbar^{1/3-\delta} e^{\pm i\pi/3}$$

and denote Γ'_\pm to be the downward gradient flowlines of $\operatorname{Re}\Phi(z)$ starting from z'_\pm ending at -1 , as shown in the middle panel of Figure 2. Then we define the deformed contour

$\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\begin{cases} \Gamma_1 = (\cup_{a=\pm} \Gamma'_a) \cap \{|z+1| < \delta\} \\ \Gamma_2 = (\cup_{a=\pm} \Gamma'_a) \cap \{|z+1| > \delta\} \\ \Gamma_3 = \{1 - r\hbar^{1/3}e^{i\pi/3} \mid r \in [1, \hbar^{-\delta}]\} \cup \{1 - r\hbar^{1/3}e^{-i\pi/3} \mid r \in [1, \hbar^{-\delta}]\} \\ \quad \cup \{1 - \hbar^{1/3}e^{i\theta} \mid \theta \in [-\pi/3, \pi/3]\} \end{cases}$$

Next, we estimate the contribution from the three parts of the contours.

On Γ_1 , the amplitude satisfies

$$|A(z)| \leq C\hbar^{-d/2}|z+1|^{-d/2},$$

while the real part of the phase obeys

$$\operatorname{Re}(\Phi(z)/\hbar) \leq -C(\hbar|z+1|)^{-1}.$$

By the Excision Lemma 3.1, the integral over Γ_1 is $O(\hbar^\infty)$.

On Γ_2 , we have

$$\sup_{z \in \Gamma_2} \operatorname{Re} \Phi(z) = \operatorname{Re} \Phi(z'_\pm) = -\frac{1}{24}\hbar^{-3\delta}(1 + O(\hbar^{2\delta}))$$

with the implied constants uniform when s varies over compact sets. Therefore, by the Excision Lemma 3.1 again, the integral over Γ_2 is $O(\hbar^\infty)$.

Finally, on Γ_3 , we note that $\hbar^{1/3} < |z-1| < \hbar^{1/3-\delta}$, hence over Γ_3 for small enough \hbar , we have

$$A(z) = (2\pi\hbar(1-z))^{-d/2}(1 + R_1(z))$$

where there exists $C_1 > 0$ such that $|R_1(z)| < C_1|z-1|$ for all $|z-1| < \hbar^{1/3-\delta}$. Hence over Γ_3 , we have

$$|(1-z)^{-d/2}R_1(z)| < C_1|z-1|^{-d/2+1} < C_1\hbar^{(-d/2+1)(1/3)}$$

Similarly, for $\Phi(z)$, we consider its Taylor expansion around $z=1$:

$$\Phi(z) = \frac{\hbar^{2/3}s}{2}(z-1) + \underbrace{\frac{\hbar^{2/3}s}{4}(z-1)^2}_{R_3(z)} + \left(-\frac{(z-1)^3}{24} + \underbrace{\frac{\hbar^{2/3}s(z-1)^3}{8}}_{R_4(z)} \right) + R_2(z)$$

where there exists $C_2 > 0$, which is uniformly bounded when u varies over a compact set, such that $|R_2(z)| < C_2|z-1|^4$ for all $|z-1| < \hbar^{1/3-\delta}$. We may rewrite the integral using $z = 1 - \hbar^{1/3}T$ and reverse the contour orientation, then

$$T \in \Gamma_T := \{re^{\pm i\pi/3} \mid r \in [1, \hbar^{-\delta}]\} \cup \{e^{i\theta/3} \mid \theta \in [-\pi/3, \pi/3]\}$$

The phase function and the amplitude become

$$-A(z(T))dz(T) = (2\pi)^{-d/2}\hbar^{1/3-2d/3}[T^{-d/2} + \tilde{A}_\hbar(T)]\frac{dT}{2\pi i}$$

where from the bound on $R_1(z)$ we have

$$|\tilde{A}_\hbar(T)| = |T^{-d/2}R_1(z(T))| \leq C_1|T|^{-d/2+1}\hbar^{1/3} < C_1\hbar^{1/3}$$

and

$$\hbar^{-1}\Phi(z(T)) = \frac{T^3}{24} - \frac{Ts}{2} + \hbar^{-1}(R_2(z(T)) + R_3(z(T)) + R_4(z(T))).$$

The remainder can be bounded as follows

$$\begin{aligned} \hbar^{-1}|R_2(z(T))| &\leq C_2 \hbar^{1/3} T^4 \\ \hbar^{-1}|R_3(z(T))| &\leq (|s|/4) \hbar^{1/3} T^2 \\ \hbar^{-1}|R_4(z(T))| &\leq (|s|/8) \hbar^{2/3} T^3. \end{aligned}$$

The above argument shows that we may apply the Trimming Lemma 3.2, to get the contribution from Γ_3 to be

$$\begin{aligned} I_3 &= \int_{\Gamma_3} A(z) e^{\Phi(z)/\hbar} dz \\ &= (2\pi)^{-d/2} \hbar^{(1-2d)/3} \int_{\mathcal{C}} T^{-d/2} e^{\frac{T^3}{24} - \frac{Ts}{2}} \frac{dT}{2\pi i} (1 + O(\hbar^{1/3})) \\ &= 2^{-d+1} \pi^{-d/2} \hbar^{(1-2d)/3} \text{Ai}_{-d/2}(s) (1 + O(\hbar^{1/3})) \end{aligned}$$

This yields the desired result for $\Pi_{\hbar}(x, x)$. The same argument applies straightforwardly to $\partial_{x_i} \Pi_{\hbar}(x, x)$, $\partial_{x_i} \partial_{y_j} \Pi_{\hbar}(x, x)$. We get

$$\begin{aligned} \Pi_{\hbar}^{-1} \partial_{x_i} \Pi_{\hbar}(x, x) &= \frac{\oint \left(-\frac{x_i}{\hbar} \frac{1-z}{1+z} \right) A(z) e^{\Phi(z)/\hbar} dz}{\oint A(z) e^{\Phi(z)/\hbar} dz} \\ &= -x_i \hbar^{-2/3} \frac{\text{Ai}_{-d/2+1}(s)}{\text{Ai}_{-d/2}(s)} (1 + O(\hbar^{1/3})) \end{aligned} \quad (35)$$

and

$$\begin{aligned} \Pi_{\hbar}^{-1} \partial_{x_i} \partial_{y_j} \Pi_{\hbar}(x, x) &= \frac{\oint \left[\frac{x_i x_j}{\hbar^2} \left(\frac{1-z}{1+z} \right)^2 + \frac{\delta_{ij}}{\hbar} \frac{2z}{1-z^2} \right] A(z) e^{\Phi(z)/\hbar} dz}{\oint A(z) e^{\Phi(z)/\hbar} dz} \\ &= \left(x_i x_j \hbar^{-4/3} \frac{\text{Ai}_{-d/2+2}(s)}{\text{Ai}_{-d/2}(s)} + \delta_{ij} \hbar^{-4/3} \frac{\text{Ai}_{-d/2-1}(s)}{2 \text{Ai}_{-d/2}(s)} \right) (1 + O(\hbar^{1/3})). \end{aligned} \quad (36)$$

Combining these, we get

$$\begin{aligned} \Omega_{ij}(u) &= \Pi_{\hbar}^{-1} \partial_{x_i} \partial_{y_j} \Pi_{\hbar}(x, x) - \Pi_{\hbar}^{-2} \partial_{x_i} \Pi_{\hbar}(x, x) \partial_{x_j} \Pi_{\hbar}(x, x) \\ &= \hbar^{-4/3} \left(\delta_{ij} \frac{\text{Ai}_{-d/2-1}(s)}{2 \text{Ai}_{-d/2}(s)} + x_i x_j \left(\frac{\text{Ai}_{-d/2+2}(s)}{\text{Ai}_{-d/2}(s)} - \frac{\text{Ai}_{-d/2+1}^2(s)}{\text{Ai}_{-d/2}^2(s)} \right) \right) (1 + O(\hbar^{1/3})). \end{aligned}$$

This completes the proof of Proposition 3.3. \square

REMARK 8. By substituting $s \mapsto \hbar^{\alpha-2/3}s$, and applying the asymptotic expansion of the weighted airy function Ai_k (see Proposition A.1), we may recover the leading term of the corresponding results on Π and Ω in the allowed and forbidden annuli. However, this does not give an estimate of the error terms. We leave the more detailed analysis for Section 6.

3.4. Proof of the off-diagonal scaling asymptotics of Theorem 1.1. To complete the proof of Theorem 1.1, we now give the full off-diagonal scaling asymptotics of the covariance function (10). The proof is a development of the diagonal result in Proposition 3.3. We fix $x_0 \in \mathcal{C}_E$ and consider

$$\Pi_{h,E}(x_0 + \hbar^{2/3}u, x_0 + \hbar^{2/3}v) = \sum_{\beta \in (\mathbb{Z}_{\geq 0})^d, |\beta|=N} \varphi_{h,\beta}(x_0 + \hbar^{3/2}u) \varphi_{h,\beta}(x_0 + \hbar^{3/2}v). \quad (37)$$

Theorem 1.1 asserts that (up to a scalar factor) the scaling limit of the kernels (37) is the kernel (5).

Proof. The proof of the off-diagonal scaling asymptotics is similar to that of the on-diagonal, so we only give a brief sketch of it.

Repeating the proof of Proposition 3.3, we again localize the integral for $\Pi_h(x, x)$ to the contour Γ_3 . We then rescale $z = 1 - \hbar^{1/3}T$ in (22) and again apply the Trimming Lemma 3.2 to obtain

$$\Pi_h(x_0 + \hbar^{2/3}u, x_0 + \hbar^{2/3}v) = (2\pi)^{-d/2} \hbar^{-2d/3+1/3} \int_{\mathcal{C}} T^{-d/2} e^{-\frac{(u-v)^2}{2T} + \frac{T^3}{24} - \frac{T}{2} \langle u+v, x_0 \rangle} dT (1 + O(\hbar^{1/3})). \quad (38)$$

On the other hand, we may rewrite (5) using Lemma A.2 on products of Airy functions. We change the integration variable $T \mapsto 2^{-4/3}T$ to get,

$$\text{Ai}(2^{1/3}(u_1 + p^2/2)) \text{Ai}(2^{1/3}(v_1 + p^2/2)) = \int_{\mathcal{C}} e^{\frac{T^3}{24} - (u_1+v_1+p^2)\frac{T}{2} - \frac{(u_1-v_1)^2}{2T}} \sqrt{\frac{1}{2\pi T}} \frac{2^{-2/3}dT}{2\pi i}$$

Substituting into (5), and computing the dp Gaussian integral, we get

$$\begin{aligned} \Pi_0(u_1, u'; v_1, v') &= (2\pi)^{(-d+1)/2} \int_{\mathcal{C}} T^{-(d-1)/2} e^{\frac{T^3}{24} - (u_1+v_1+p^2)T/2 - \frac{(u_1-v_1)^2}{2T} - \frac{(u'-v')^2}{2T}} \sqrt{\frac{1}{2\pi T}} \frac{dT}{2\pi i} \\ &= (2\pi)^{-d/2} \int_{\mathcal{C}} T^{-d/2} e^{\frac{T^3}{24} - (u_1+v_1)\frac{T}{2} - \frac{(u-v)^2}{2T}} \frac{dT}{2\pi i} \end{aligned} \quad (39)$$

This agrees with the right hand side of (38), hence gives the proof for off-diagonal formula (4). This combines with the diagonal result (33) in Proposition 3.3 finishes the proof of Theorem 1.1. \square

REMARK 9. We may also easily derive the result of Ω_{ij} in the $\alpha = 2/3$ region from the off-diagonal scaling limit of Π_h .

$$\begin{aligned} \Omega_{ij}(x_0 + \hbar^{2/3}u_0) &= \hbar^{-4/3} \partial_{u_i} \partial_{v_j} |_{u=v=u_0} \log(\Pi_h(x_0 + \hbar^{2/3}u, x_0 + \hbar^{2/3}v)) \\ &= \hbar^{-4/3} \partial_{u_i} \partial_{v_j} |_{u=v=u_0} \log(\Pi_0(u, v))(1 + O(\hbar^{1/3})) \end{aligned}$$

which immediately gives the correct scaling law of $\Omega \sim \hbar^{-4/3}$. For the full Ω_{ij} , one can use the contour integral expression (39) for Π_0 , then using the definition for the weighted Airy function. It is the same calculation as (35) and (36).

3.5. L^2 mass near the caustic: Proof of Corollary 1.2.

Proof of Corollary 1.2. A random L^2 -normalized eigenfunction $\Psi_{h,E}$ is distributed according to the uniform measure on the unit sphere from $V_{h,E}$. Hence,

$$\Psi_{h,E} = \sum_{|\beta|=N} \frac{a_\beta}{|\vec{a}|} \phi_{\beta,h}$$

where $\vec{a} = (a_\beta)$ is a standard Gaussian vector and $|\vec{a}| = \left(\sum_\beta |a_\beta|^2\right)^{1/2}$ is its length. Write $\vec{\phi} = (\phi_\beta)_\beta$. Then

$$\mathbf{E}|\Psi(x)|^2 = \mathbf{E} \left| \left\langle \frac{\vec{a}}{|\vec{a}|}, \vec{\phi}(x) \right\rangle \right|^2 = |\vec{\phi}(x)|^2 \mathbf{E} \left| \left\langle \frac{\vec{a}}{|\vec{a}|}, \frac{\vec{\phi}(x)}{|\vec{\phi}(x)|} \right\rangle \right|^2 = \frac{\Pi_h(x, x)}{\dim V_{h,E}}.$$

We then integrate (6) of Theorem 1.1 over $T_\delta(\mathcal{C}_E)$ and use (8) and the equation for weighted Airy functions in the second line of (59) of Appendix §A to find that

$$\begin{aligned} \mathbf{E} M_2(\Psi_{h,E}, T_\delta(\mathcal{C}_E)) &= \frac{1}{\dim V_{h,E}} \int_{T_\delta(\mathcal{C})} \Pi_{h,E}(x, x) dx \\ &\approx \Gamma(d)(2h)^{d-1} \int_{\mathcal{C}} \int_{-\kappa}^{\kappa} \Pi_{h,E}(x_0(1 + \hbar^{2/3}r), x_0(1 + \hbar^{2/3}r)) \hbar^{2/3} dr dS(x_0) \\ &= \Gamma(d)(2h)^{d-1} \cdot \frac{2\pi^{d/2}}{\Gamma(d/2)} \hbar^{2/3} \int_{-\kappa}^{\kappa} \Pi_{h,E}(x_0(1 + \hbar^{2/3}r), x_0(1 + \hbar^{2/3}r)) dr \\ &\approx \Gamma(d)(2h)^{d-1} \cdot \frac{2\pi^{d/2}}{\Gamma(d/2)} \hbar^{2/3} \int_{-\kappa}^{\kappa} 2^{-d+1} \pi^{-d/2} \hbar^{(1-2d)/3} \text{Ai}_{-d/2}(2r) dr \\ &= \frac{\Gamma(d)}{\Gamma(d/2)} \hbar^{d/3} \int_{-2\kappa}^{2\kappa} \text{Ai}_{-d/2}(s) ds \\ &= \frac{\Gamma(d)}{\Gamma(d/2)^2} \hbar^{d/3} \int_{-2\kappa}^{2\kappa} \int_0^\infty \text{Ai}(s + \rho) \rho^{d/2-1} d\rho ds \end{aligned}$$

which gives the stated result with $C(d) = \frac{\Gamma(d)}{\Gamma(d/2)^2}$. □

3.6. Scaling limit random Wave Ensemble near the Caustic. The scaled kernel (5) is the covariance kernel for a limiting (infinite dimensional) ensemble of Gaussian random functions on $\mathbb{R}^d \simeq T_{x_0}\mathcal{C}_E \oplus N_{x_0}\mathcal{C}_E$, where $N_{x_0}\mathcal{C}_E$ is the fiber of the normal bundle to \mathcal{C}_E at x_0 . This scaled covariance function corresponds to a Hilbert space of functions on \mathbb{R}^d obtained as scaling limits of Hermite eigenfunctions in the eigenspaces $V_{h,E}$.

To show this explicitly, consider the eigenfunctions

$$\hat{H}_h \psi_{h,E} = E \psi_h.$$

We rescale this equation around $x_0 \in \mathcal{C}_E$ using the local dilation operator

$$D_h^{x_0} \psi(u) = \psi(x_0 + \hbar^\alpha u).$$

The equation above then is equivalent to

$$D_h^{x_0} \circ \hat{H}_h \circ (D_h^{x_0})^{-1} \circ D_h^{x_0} \psi_{h,E} = E D_h^{x_0} \psi_h.$$

Now,

$$D_h^{x_0} \circ \widehat{H}_h \circ (D_h^{x_0})^{-1} = -\frac{1}{2} \hbar^2 \hbar^{-2\alpha} \Delta_u + \frac{1}{2} |x_0 + \hbar^\alpha u|^2 = -\frac{1}{2} \hbar^{2-2\alpha} \Delta_u + \frac{|x_0|^2}{2} + \hbar^\alpha \langle x_0, u \rangle + R(\hbar, x_0, u).$$

Note that since $|x_0|^2/2 = V(x_0) = E$, the terms $ED_h^{x_0} \psi_h$ cancel. The eigenvalue equation becomes a harmonic equation

$$(-\hbar^{2-2\alpha} \Delta_u + 2\hbar^\alpha \langle x_0, u \rangle + R(\hbar, x_0, u)) \psi_h^{x_0}(u) = 0.$$

The equation is a small perturbation of the *osculating equation*

$$(-\hbar^{2-2\alpha} \Delta_u + 2\hbar^\alpha \langle x_0, u \rangle) \psi_h^{x_0}(u) = 0.$$

If $\alpha = \frac{2}{3}$ all factors of \hbar may be eliminated from the osculating equation, giving

$$(\Delta_u - 2u_1) \psi_h^{x_0}(u) = 0.$$

Here we choose coordinates (u_1, u') so that $\langle x_0, u \rangle = u_1$, i.e. $x_0 = (1, 0, \dots, 0)$.² The osculating equation is separable, and becomes

$$\left(\left(\frac{\partial^2}{\partial u_1^2} - 2u_1 \right) + \Delta_{u'} \right) \psi^{x_0} = 0. \quad (40)$$

We write

$$\psi^{x_0}(u_1, u') = f(u_1)g(u')$$

to get on $\mathbb{R}_{u_1} \times \mathbb{R}_{u'}^{d-1}$,

$$\left(\frac{\partial^2}{\partial u_1^2} - 2u_1 \right) f(u_1) = \lambda f(u_1), \quad \Delta_{u'} g(u') = -\lambda g(u'). \quad (41)$$

We define \mathcal{H}_∞ be the space of temperate solutions of (40), i.e. solutions in $\mathcal{S}'(\mathbb{R}^d)$.

The temperate eigenfunctions of $y'' - xy = \lambda y$ (i.e. lying in $\mathcal{S}'(\mathbb{R})$) are $\{\text{Ai}(x + \lambda)\}$. The spectrum of the Airy operator is purely absolutely continuous with multiplicity one on all of \mathbb{R} ([G, T] and [O] (Chapter 6)). Note that $y'' - xy = \lambda u$ is equivalent to $y'' - \tilde{x}y = 0$ if $\tilde{x} = x - \lambda$. It follows that a basis of temperate solutions of (40) are product solutions

$$(2\pi)^{(1-d)/2} \text{Ai}(2^{1/3}(u_1 + p^2/2)) 2^{1/3} e^{i\langle p, u' \rangle}, \quad p \in \mathbb{R}^{d-1}, \quad \lambda = p^2. \quad (42)$$

There is a natural isomorphism $W : L^2(\mathbb{R}^{d-1}) \rightarrow \mathcal{H}_\infty$, which can be used to endow \mathcal{H}_∞ with an inner product. Taking the Fourier transform of the osculating equation gives

$$\left(-|\xi'|^2 - |\xi_1|^2 + \frac{2}{i} \frac{\partial}{\partial \xi_1} \right) \widehat{\psi} = 0.$$

This is a first order linear equation with “time parameter” ξ_1 , and we write $\widehat{\psi} = \widehat{\psi}(\xi_1, \xi')$. Then the Cauchy problem

$$\begin{cases} \frac{1}{i} \frac{\partial}{\partial \xi_1} \widehat{\psi}(\xi_1, \xi') = \frac{1}{2} (|\xi'|^2 + |\xi_1|^2) \widehat{\psi}(\xi_1, \xi') \\ \psi(0, \xi') = \widehat{\psi}_0(\xi') \end{cases}$$

is solved by the unitary propagator $U(\xi_1)$ on $L^2(\mathbb{R}^{d-1}, d\xi')$ defined by

$$\widehat{\psi}(\xi', \xi_1) = U(\xi_1) \widehat{\psi}_0(\xi') = e^{\frac{i}{2}(\xi_1 |\xi'|^2 + \xi_1^3/3)} \widehat{\psi}_0(\xi'). \quad (43)$$

²Here, as above, we set $E = 1/2$ as explained near Eq (17).

It follows that \mathcal{H}_∞ is isomorphic to the space of Cauchy data $\widehat{\psi}_0$.

LEMMA 3.4. *Let $g \in L^2(\mathbb{R}^{d-1}, dx)$ and let $G(\xi_1, \xi') := U(\xi_1)g(\xi')$. Also, let \mathcal{F}^* denote the inverse Fourier transform on \mathbb{R}^d . Then the linear isomorphism*

$$\begin{aligned} Wg(u_1, u') = [\mathcal{F}^*G](u_1, u') &= \int_{\mathbb{R}^{d-1} \times \mathbb{R}} e^{i\langle u', p \rangle} e^{i\xi_1 u_1} e^{\frac{i}{2}(\xi_1 |p|^2 + \xi_1^3/3)} g(p) d\xi_1 dp \\ &= \int_{\mathbb{R}^{d-1}} e^{i\langle u', p \rangle} \text{Ai}(2^{1/3}(u_1 + |p|^2/2)) 2^{1/3} g(p) dp. \end{aligned} \quad (44)$$

maps $L^2(\mathbb{R}^{d-1}) \rightarrow \mathcal{H}_\infty$ bijectively.

Proof. W is obviously injective and takes its values in \mathcal{H}_∞ . To prove surjectivity, we let

$$g_{v_1, v'}(p) = (2\pi)^{-(d-1)/2} e^{-i\langle v', p \rangle} \text{Ai}(2^{1/3}(v_1 + |p|^2/2)) 2^{1/3} \in L^2(\mathbb{R}^{d-1}, dx).$$

Explicitly,

$$\|g_{v_1, v'}\|_{L^2(\mathbb{R}^{d-1})}^2 = C_d \int_0^\infty |\text{Ai}(2^{1/3}(v_1 + \rho))|^2 \rho^{\frac{d-1}{2}} d\rho.$$

for some constant C_d . The integral converges since $|\text{Ai}(2^{1/3}(v_1 + \rho))|^2$ decays exponentially as $\rho \rightarrow \infty$ in \mathbb{R}_+ . We then observe that

$$Wg_{v_1, v'}(u_1, u') = \Pi_0(u_1, u', v_1, v'). \quad (45)$$

But as observed above, product solutions (42) span \mathcal{H}_∞ and we obtain all of them in (44).

The inverse W^{-1} can be explicitly described as follows: the range of the Fourier transform restricted to \mathcal{H}_∞ ,

$$\mathcal{F} : \mathcal{H}_\infty \rightarrow \mathcal{S}'(\mathbb{R}^d)$$

is the subspace of temperate functions satisfying the functional equation (43). Thus

$$W^{-1} : \psi \in \mathcal{H}_\infty \rightarrow \widehat{\psi}(0, \xi')$$

is an injective map to $L^2(\mathbb{R}^d)$ which inverts W . We may write $W^{-1} = \mathcal{R}\mathcal{F}$ where $\mathcal{R}F(\xi') = F(0, \xi')$. \square

DEFINITION 3.5. *We define an inner product on \mathcal{H}_∞ by*

$$\langle Wg, Wh \rangle_{\mathcal{H}_\infty} := \langle g, h \rangle_{L^2(\mathbb{R}^{d-1})}.$$

LEMMA 3.6. *With the above inner product, $\Pi_0^2 = \Pi_0$.*

Proof. By (44),

$$\begin{aligned} \Pi_0^2(u_1, u', v_1, v') &= \langle Wg_{u_1, u'}, Wg_{v_1, v'} \rangle \\ &= \langle g_{u_1, u'}, g_{v_1, v'} \rangle_{L^2(\mathbb{R}^{d-1})} \\ &= \Pi_0(u_1, u', v_1, v'). \end{aligned}$$

\square

We also could use Π_0 directly to define an inner product on \mathcal{H}_∞ using the method of reproducing kernel Hilbert spaces. According to the Aronszajn theorem, a symmetric and positive definite kernel defines a unique reproducing kernel Hilbert space (RKHS) [A]. We briefly recall that a kernel $K(x, y)$ on a space $X \times X$ is called positive-definite if $(K(x_i, x_j))_{i,j \leq M}$ defines a positive Hermitian matrix. Since Π_0 is the limit of positive definite kernels, it is positive definite and therefore induces an inner product on \mathcal{H}_∞ . We claim that the RKHS is the same \mathcal{H}_∞ equipped with the inner product of Definition 3.5.

By definition, the RKHS associated to Π_0 is the closure of the set of functions of the form

$$\left\{ g_{v_1, v', \vec{a}}(\cdot) = \sum_{j=1}^n a_j \Pi_0(\cdot, v_1, v'), \quad a_j \in \mathbb{R} \right\},$$

equipped with the inner product

$$\langle g_{v_1, v', \vec{a}}, g_{u_1, u', \vec{b}} \rangle_H = \sum_{j,k=1}^n a_j b_k \Pi_0(u_1, u', v_1, v'),$$

and it follows by (45) and Lemma 3.6 that this inner product is the same as Definition 3.5.

The scaled density of the random nodal set in Theorem 1.4 can be identified as the density of zeros of the Gaussian random functions in \mathcal{H}_∞ .

4. COMPLETION OF THE PROOF OF THEOREM 1.4

The formulae (34) give scaling asymptotics for the entries of the Kac-Rice matrix of Lemma 2.2. As mentioned in the remark after the statement of Theorem 1.4, we still need to prove the positivity of the first term in our expansion of the spectral projector near the caustic in Theorem 1.1. The proof is supplied by

PROPOSITION 4.1. $\text{Ai}_{-d/2}(s) > 0$ for all integers $d \geq 2$ and $s \in \mathbb{R}$.

Proof. By Proposition A.1, as $s \rightarrow \infty$, $\text{Ai}_k(s) \rightarrow 0$. Also note that

$$\frac{d}{ds} \text{Ai}_k(s) = -\text{Ai}_{k+1}(s)$$

we know that

$$\text{Ai}_k(s) = \int_s^\infty \text{Ai}_{k+1}(s') ds'.$$

Hence, it suffices to show that $\text{Ai}_{-1}(s) > 0$ and $\text{Ai}_{-3/2}(s) > 0$ for all $s \in \mathbb{R}$.

For $\text{Ai}_{-3/2}(s)$, we use Lemma A.2 and set $x = y$ to get

$$\text{Ai}_{-1/2}(s) = \sqrt{2\pi} 2^{1/6} \text{Ai}^2(2^{-2/3}s) \geq 0$$

and integrate to get $\text{Ai}_{-3/2}(v) > 0$ for all $v \in \mathbb{R}$.

For $\text{Ai}_{-1}(s)$, we use the fact that $\text{Ai}_0(s) = \text{Ai}(s)$ and the first zero of $\text{Ai}(s)$ is at $s \approx -2.8$ to see $\text{Ai}_{-1}(s) > 0, \forall s \geq -2$. We now show that $\text{Ai}_{-1}(s) > c > 0$ when $s < -2$ as well. We use the method of stationary phase, following [BPD]. We first deform the contour integral \mathcal{C} into union of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, see Figure 3. Here and below we use the same notation for a contour \mathcal{C}_j and the integral over the contour.

The contribution from \mathcal{C}_1 to $\text{Ai}_{-1}(s)$ for $s < -2$ is

$$\mathcal{C}_1 = \frac{1}{2\pi i} \int_{\mathcal{C}_1} e^{\frac{T^3}{3} + T|s|} \frac{dT}{T} = \frac{1}{2\pi i} \oint_{|T|=\epsilon} e^{\frac{T^3}{3} + T|s|} \frac{dT}{T} = 1.$$

The contribution from \mathcal{C}_2 and \mathcal{C}_3 are complex conjugates of each other, hence it suffices to compute the real part of one of them. Here we deform the contour \mathcal{C}_2 again for computing the upper bound. The sum of the contributions from \mathcal{C}_2 and \mathcal{C}_3 is

$$\begin{aligned} \mathcal{C}_2 + \mathcal{C}_3 &= 2\text{Re} \left\{ \frac{1}{2\pi i} \int_{\mathcal{C}_2} e^{\frac{T^3}{3} + T|s|} \frac{dT}{T} \right\} \\ &= 2\text{Re} \left\{ \frac{1}{2\pi i} \left(\int_{T=-\infty}^{-\sqrt{|s|}} + \int_{T=\sqrt{|s|}e^{i\theta}, \theta \in (\pi, \pi/2)} + \int_{T=\sqrt{|s|}(i + \exp(i\pi/4)\rho)} \right) e^{\frac{T^3}{3} + T|s|} \frac{dT}{T} \right\} \\ &= 2\text{Re} \left\{ \frac{-1}{2\pi i} \int_{\sqrt{|s|}}^{\infty} e^{-\rho^3/3 - \rho|s|} \frac{d\rho}{\rho} \right\} + 2\text{Re} \left\{ \frac{1}{2\pi} \int_{\pi}^{\pi/2} e^{|s|^{3/2}(e^{3i\theta}/3 + e^{i\theta})} d\theta \right\} \\ &\quad + 2\text{Re} \left\{ \frac{1}{2\pi i} \int_0^{\infty} e^{-|s|^{3/2}\rho^2 - \frac{|s|^{3/2}}{3\sqrt{2}}\rho^3} e^{i(2/3)|s|^{3/2} + i/(3\sqrt{2})|s|^{3/2}\rho^3} \frac{d\rho}{e^{i\pi/4} + \rho} \right\} \\ &=: K_1 + K_2 + K_3. \end{aligned}$$

K_1 is the real part of a purely imaginary number, hence $K_1 = 0$. For K_2 , we bound the integrands by the sup-norm hence

$$K_2 = 2\text{Re} \frac{1}{2\pi} \int_{\pi}^{\pi/2} e^{|s|^{3/2}(e^{3i\theta}/3 + e^{i\theta})} d\theta \geq -\frac{1}{\pi} \int_{\pi}^{\pi/2} \left| e^{|s|^{3/2}(e^{3i\theta}/3 + e^{i\theta})} \right| d\theta \geq -\frac{1}{\pi} \int_{\pi}^{\pi/2} d\theta = -0.5,$$

where we used the fact that $\cos(3\theta)/3 + \cos(\theta) \leq 0$ for $\theta \in [\pi/2, \pi]$. And for K_3 we have

$$\begin{aligned} K_3 &\geq -\frac{1}{\pi} \left| \int_0^{\infty} e^{-|s|^{3/2}\rho^2 - \frac{|s|^{3/2}}{3\sqrt{2}}\rho^3} e^{i\frac{2}{3}|s|^{3/2} + i|s|^{3/2}\rho^3/(3\sqrt{2})} \frac{d\rho}{e^{i\pi/4} + \rho} \right| \\ &\geq -\frac{1}{\pi} \int_0^{\infty} e^{-|s|^{3/2}\rho^2} d\rho = -\frac{1}{2\sqrt{\pi}|s|^{3/4}} \geq -\frac{1}{2\sqrt{\pi}2^{3/4}} \geq -0.2 \end{aligned}$$

Thus, we get $\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 \geq 0.3$ when $s < -2$, hence $\text{Ai}_{-1}(s) > 0$ for all $s \in \mathbb{R}$. □

5. INTERSECTIONS OF THE NODAL SET WITH THE CAUSTIC: PROOF OF THEOREM 1.6

Theorem 1.6 follows from our formulae for (10) restricted to the caustic together with the Kac-Rice formula for the expected number of intersections of the nodal set with the caustic. The result is analogous to the formula in [TW] (Proposition 3.2) for the expected number of intersections of nodal lines with the boundary of a plane domain. The argument that the Kac-Rice formula can be applied to measure the volume of nodal intersections with the caustic is identical to that given in the beginning of §??.

Proof. In the notation of the previous section, with $x = x_0 + \hbar^{2/3}u$, with $x_0^2 = 1$, $x_0 = (1, 0, \dots, 0)$ and $s = 2\langle x_0, u \rangle$, the restriction to the caustic is $u = 0$ and therefore $s = 0$.

The covariance matrix used in the Kac-Rice formula is

$$\begin{aligned}\Omega_{ij}(x_0) &= (\partial_{u_i} \partial_{v_j} |_{u=v} \log \Pi_{h,E}(x_0 + u\hbar^{2/3}, x_0 + v\hbar^{2/3}))_{2 \leq i, j \leq d} \\ &= \hbar^{-4/3} \frac{\delta_{ij} \operatorname{Ai}_{-1-d/2}(0) \operatorname{Ai}_{-d/2}(0)}{2 \operatorname{Ai}_{-d/2}^2(0)} (1 + O(\hbar^{1/3})) = \hbar^{-4/3} \frac{\delta_{ij} \operatorname{Ai}_{-1-d/2}(0)}{2 \operatorname{Ai}_{-d/2}(0)} (1 + O(\hbar^{1/3})).\end{aligned}\tag{46}$$

We calculate the constant $\mathcal{F}_{C_E, d}$ as follows:

$$\begin{aligned}\mathcal{F}_{C_E, d} &= (2\pi)^{-d/2} \int_{\mathbb{R}^{d-1}} |\xi| \left(\frac{\operatorname{Ai}_{-1-d/2}(0)}{2 \operatorname{Ai}_{-d/2}(0)} \right)^{1/2} e^{-\xi^2/2} d\xi \\ &= (2\pi)^{-1/2} \left(\frac{\operatorname{Ai}_{-1-d/2}(0)}{2 \operatorname{Ai}_{-d/2}(0)} \right)^{1/2} \int_{\mathbb{R}^{d-1}} |\xi| e^{-\xi^2/2} \frac{d\xi}{(2\pi)^{(d-1)/2}} \\ &= (2\pi)^{-1/2} \left(\frac{\operatorname{Ai}_{-1-d/2}(0)}{\operatorname{Ai}_{-d/2}(0)} \right)^{1/2} \frac{\Gamma(d/2)}{\Gamma((d-1)/2)}\end{aligned}$$

In the case of $d = 2$, we have

$$\mathcal{F}_{C_E, 2} = \frac{1}{\sqrt{2\pi}} \left(\frac{\operatorname{Ai}_{-2}(0)}{\operatorname{Ai}_{-1}(0)} \right)^{1/2},$$

and hence

$$\mathbf{E} (\#Z_{\Phi_{h,E}} \cap \mathcal{C}_E) = \hbar^{-2/3} \mathcal{F}_{C_E, 2} (2\pi) = \hbar^{-2/3} \sqrt{2} \left(\frac{\operatorname{Ai}_{-2}(0)}{\operatorname{Ai}_{-1}(0)} \right)^{1/2}.$$

This concludes the proof. \square

6. ALLOWED AND FORBIDDEN ANNULI FOR $\alpha < \frac{2}{3}$: PROOF OF THEOREM 1.5

The main results of this section are Propositions 6.1 and 6.5, giving asymptotic formulae for the Kac-Rice matrix in h^α tubes around the caustic where $0 < \alpha < 2/3$. In Section 6.1 we find the asymptotics in the allowed region for $\alpha < 2/3$. In Section 6.5 we do the same in the forbidden region.

REMARK 10. *Before going into the details of the proofs, we explain how the asymptotics for $0 < \alpha < \frac{2}{3}$ are related to those for $\alpha = 0$ or $\alpha = 2/3$. The asymptotics have a leading term and a remainder term. The leading term for $0 < \alpha < \frac{2}{3}$ can be formally obtained by interpolation from the leading term of the $\alpha = 0$ result or the $\alpha = 2/3$ result. But this would not prove that the asymptotics are valid, because one still has to prove that the remainder term for $0 < \alpha < \frac{2}{3}$ is smaller than the purported leading term. The proof we give uses the stationary phase approach similarly to $\alpha = 0$ but with a nearly degenerate quadratic phase function and keeps track of how the degeneracy affects the remainder estimate.*

6.1. The allowed annuli region.

PROPOSITION 6.1. *Let $|x|^2 = 1 - \hbar^\alpha s$ for any $s > 0, \alpha \in (0, 2/3)$. We have*

$$\Pi_h(x, x) = C_d \hbar^{\Delta(\alpha, d)} s^{d/2-1} (1 + O(\hbar^{1-\frac{3}{2}\alpha}) + O(\hbar^{\alpha/2})),$$

where

$$\Delta(\alpha, d) = 1 - d + \frac{d-2}{2}\alpha \quad \text{and} \quad C_d = \frac{2^{-d+1}\pi^{-d/2}}{\Gamma\left(\frac{d}{2}\right)}.$$

Further,

$$\Omega_{ij}(x) = d^{-1}\delta_{ij}\hbar^{-2+\alpha}s(1 + O(\hbar^{1-(3/2)\alpha}) + O(\hbar^{\alpha/2})).$$

The implied constants in $O(\cdots)$ are uniform for s in a compact subset of $(0, \infty)$.

Proof. We give the details for the calculation for $\Pi_{\hbar}(x, x)$. The outline of the proof is given in Section 3.1 and our starting point is (27). As explained in Section 3.1, the integrand $A(z)e^{\Phi(z)/\hbar}$ is defined on

$$S = \mathbb{C} \setminus ((\infty, -1] \cup [1, \infty) \cup \{0\}).$$

Since the integrand is holomorphic, we may deform the contour $C_{\epsilon} = \{|z| = 1 - \epsilon\}$ within S . For the portion near -1 , we break C_{ϵ} into two pieces, one terminating at $-1 + \epsilon + i0$ and the other starting at $-1 + \epsilon - i0$. Moreover, since the integrand in (27) is rapidly decaying at $\infty \pm i\tau$ for every $\tau > 0$, we may deform C_{ϵ} all way to $\infty \pm i0$ inside S (as shown in Figure 2). Finally, since $z_{\pm} \in S$, we can make the deformed contour pass through z_{\pm} , and in a $\hbar^{\alpha/2}$ neighborhood be given by

$$z_{\pm} + \hbar^{\alpha/2}\eta \left(1 \mp \frac{i}{\sqrt{3}}\right), \quad \eta \in \mathbb{R}. \quad (47)$$

This completes step (i) of the proof outline from Section 3.1. We have

$$\int_{|z|=1-\epsilon} A(z)e^{\Phi(z)/\hbar} dz = \sum_{\pm} \underbrace{\int_{C_{\pm}} A(z)e^{\Phi(z)/\hbar} dz}_{I_{\pm}} + \underbrace{\int_{C_0} A(z)e^{\Phi(z)/\hbar} dz}_{I_0},$$

where C_0 is the keyhole contour that starts at $\infty + i0$, wraps around 1 and ends at $\infty - i0$ and C_{\pm} are the complex conjugate contours on which $\pm \Im z > 0$ shown in the left panel of Figure 2. The purpose of requiring (47) is that

$$\operatorname{Re} \Phi \left(z_{+} + \hbar^{\alpha/2}\eta \left[1 - \frac{i}{\sqrt{3}}\right] \right) = -\hbar^{3\alpha/2} \frac{\eta^2 \sqrt{s}}{2\sqrt{3}} + O(\hbar^{2\alpha}).$$

Therefore, there exists $K > 0$ and $C > 0$ so that for all s in a compact subset of \mathbb{R}_{+}

$$\sup_{z \in C_0, C_{\pm} |z - z_{\pm}| \geq K\hbar^{\alpha/2}} \operatorname{Re} \Phi(z) < -C\hbar^{3\alpha/2}. \quad (48)$$

This ensures that $\exp(\Phi(z)/\hbar) = O(\hbar^{\infty})$ uniformly for $z \in C_0, C_{\pm}$, satisfying $|z - z_{\pm}| \geq K\hbar^{\alpha/2}$. It will turn out that I_0 grows more rapidly as $\hbar \rightarrow 0$ than I_{\pm} . The exact growth rate in \hbar of I_{\pm} is given in the following Lemma.

LEMMA 6.2. *There exists a constant $C > 0$ (depending on u) so that as $\hbar \rightarrow 0$*

$$|I_{\pm}| = C \cdot \hbar^{\delta(\alpha, d)} \left(1 + O(\hbar^{1-3\alpha/2}) + O(\hbar^{\alpha/2})\right), \quad (49)$$

where

$$\delta(\alpha, d) = \frac{1}{2} \left(1 - \frac{\alpha}{2} - d \left(1 + \frac{\alpha}{2}\right)\right).$$

Proof. It suffices to consider I_+ since $I_+ = \overline{I_-}$. We have by (30),

$$z_+ = -1 + 2|x|^2 + i2|x|\sqrt{1-|x|^2} = 1 + i \cdot 2\hbar^{\alpha/2}\sqrt{s} + O(\hbar^\alpha)$$

and $\operatorname{Re}\Phi(z_+) = 0$ since $\operatorname{Re}(\Phi)$ vanishes on the entire unit circle (except at the point $z = -1$). Recall the constant K from (48). By (48), we have $I_+ = I'_+ + O(\hbar^\infty)$, where

$$I'_+ = \int_{C_+} e^{\Phi(z)/\hbar} A(z) \chi_K(z) dz$$

and

$$\chi_K(z) = \begin{cases} 1, & |z - z_+| \leq \frac{K}{2}\hbar^{\alpha/2} \\ 0, & |z - z_+| > K\hbar^{\alpha/2} \end{cases}.$$

To evaluate the localized integral, we seek to apply the method of stationary phase (Proposition B.1) to I_+ , and we need the following Lemma to ensure that the error terms are uniformly bounded.

LEMMA 6.3. *There exists a $K > 0$ and $c, C_j > 0$, $j = 1, 2, 3, 4$, so that*

$$\inf_{z \in \operatorname{supp}(\chi_K) \cap C_+} |\Phi''(z)| > c\hbar^{\alpha/2}. \quad (50)$$

Further, for $j = 1, 2, 3$ we have

$$\sup_{z \in \operatorname{supp}(\chi_K) \cap C_+} |\partial_z^j \Phi(z)| \leq C_j \hbar^{(3-j)\frac{\alpha}{2}} \quad (51)$$

while for $j = 4$

$$\sup_{z \in \operatorname{supp}(\chi_K) \cap C_+} |\partial_z^j \Phi(z)| \leq C_j. \quad (52)$$

Proof. The estimates (50), (51), and (52) are all obtained by Taylor expansion using that

$$\Phi'(z_+) = 0 \quad \text{and} \quad \Phi''(z_+) = 2i\hbar^{\alpha/2}\sqrt{u} (1 + O(\hbar^{\alpha/2}))$$

as well as $\sup_{z \in \operatorname{supp}(\chi_K)} |\Phi^{(m)}(z)| \leq \kappa$ for some κ independent of K and $m = 3, 4$. \square

We change variables

$$z = z_+ + \hbar^{\alpha/2}\eta,$$

and \tilde{I}'_+ becomes

$$\hbar^{\alpha/2} e^{\Phi(z_+)/\hbar} \cdot \int_{\tilde{C}_+} \tilde{\psi}(s) \tilde{\chi}_K(\eta) \tilde{A}(\eta) \exp \left[\hbar^{-1+3\alpha/2} \cdot \tilde{\Phi}(\eta) \right] d\eta,$$

where $\tilde{\chi}_K(\eta) = \chi_K(z_+ + \hbar^{\alpha/2}\eta)$ and we have written

$$\tilde{\Phi}(\eta) := \hbar^{-3\alpha/2} (\Phi(z_+ + \hbar^{\alpha/2}\eta) - \Phi(z_+))$$

as well as

$$\tilde{A}(\eta) := A(z_+ + \hbar^{\alpha/2}\eta).$$

By (51) and (52), there exists $C_j > 0$ so that the phase function $\tilde{\Phi}$ satisfies

$$\sup_{\eta \in \operatorname{supp}(\tilde{\chi})} |\partial_\eta^j \tilde{\Phi}(\eta)| \leq C_j \quad (53)$$

for all h sufficiently close to 0. Moreover by (50), there exists $c > 0$ so that

$$\inf_{z \in \text{supp}(\chi)} \left| \partial_s^2 \tilde{\Phi}(\eta) \right| \geq c \quad (54)$$

for all h sufficiently close to 0. This shows that the constant C in the error term from applying stationary phase (Proposition B.1) is independent of h . We have

$$|I_+| = \frac{\hbar^{\alpha/2} |A(z_+) e^{\Phi(z_+)/\hbar}|}{\sqrt{2\pi} (|\tilde{\Phi}''(0)| / \hbar^{1-3\alpha/2})} (1 + R_+) + O(\hbar^\infty)$$

To compute the leading order term, we note that

$$\Phi(z_+) \in i\mathbb{R}, \quad |\Phi''(z_+)| = 2\hbar^{\alpha/2} \sqrt{s} (1 + O(\hbar^{\alpha/2})).$$

Thus $|e^{\Phi(z_+)/\hbar}| = 1$. Since $A(z)$ has the pole contribution $(z-1)^{-d/2}$ and $z_+ = 1 + 2i\hbar^{\alpha/2}s$ we have

$$A(z_+) = C \cdot \hbar^{-(d/2)(1+\alpha/2)} s^{-d/2} (1 + O(\hbar^{\alpha/2})).$$

Moreover, derivatives of the amplitude $\tilde{\chi}_K$ in the η variable are all bounded and for each $j \geq 0$ there exists $C_j > 0$ so that

$$\partial_\eta^j (\tilde{\chi}_K(\eta) \cdot \tilde{A}(\eta)) \leq C_j |A(z_+)|.$$

Therefore, there exists $C > 0$ so that

$$|R_+| \leq C \hbar^{1-3\alpha/2}.$$

Combining the above estimate, we have

$$|I_+| = C \hbar^{\delta(\alpha,d)} (1 + O(\hbar^{\alpha/2}) + O(\hbar^{1-3\alpha/2})),$$

completing the proof of Lemma 6.2. \square

We study I_0 as $\hbar \rightarrow 0$ in the next Lemma.

LEMMA 6.4. *As $\hbar \rightarrow 0$,*

$$I_0 = \frac{2^{1-d} \pi^{-d/2} s^{d/2-1}}{\Gamma(\frac{d}{2})} \cdot \hbar^{\Delta(\alpha,d)} (1 + O(\hbar^{2-3\alpha}) + O(\hbar^{1-\alpha})),$$

where $\Delta(\alpha, d) = -\frac{d}{2} + 1 - \alpha - \frac{d}{2}(1 - \alpha)$.

Proof. We introduce a new variable $w = (1-z)/(1+z)$, then $z = (1-w)/(1+w)$. We will abuse notations to mean $A(w) := A(z(w))$, $\Phi(w) := \Phi(z(w))$. Then we get

$$A(w) \frac{dz}{dw} = \frac{-1}{\pi i} \frac{(1-w^2)^{d/2-1}}{(4\pi\hbar)^{d/2} (1+w)^2} w^{-d/2}, \quad \Phi(w) = \hbar^\alpha s w + \sum_{j \geq 3 \text{ odd}} w^j / j,$$

In the new variable w , the contour C_0 starts from $w = -1+i0$, wraps around $[-1, 0]$ (which is the image of the branch cuts from the z -plane) counterclockwise and returns to $w = -1-i0$. We deform C_0 to

$$C_{0,-} = [-1-i0, -\hbar^{1-\alpha}/u - i0], \quad C_{0,\text{circle}} = \{|w| = \hbar^{1-\alpha}/u\} \quad C_{0,+} = [-\hbar^{1-\alpha}/u + i0, -1+i0]$$

and correspondingly write

$$I_0 = I_{0,-} + I_{0,\text{circle}} + I_{0,+}.$$

For any $\epsilon > 0$, if $w < -\hbar^{1-\alpha-\epsilon}$, then the phase $\Phi(w)$ is exponentially large in \hbar and negative so that

$$I_0 = \int_{C_0 \cap |w| > \hbar^{1-\alpha-\epsilon}} A(w) e^{\Phi(w)/\hbar} dw = O(\hbar^\infty).$$

Let us denote by $\tilde{C}_{0,\hbar}$ a contour running from $-\hbar^{-\epsilon} - i0$ counterclockwise around $(-\infty, 0]$ to $-\hbar^{-\epsilon} + i0$. Changing variables $W = sw\hbar^{\alpha-1}$ and Taylor expanding, we have

$$\begin{aligned} I_0 &= 2^{1-d}\pi^{-d/2}\hbar^{\Delta(\alpha,d)}s^{d/2-1} \cdot \frac{i}{2\pi} \int_{\tilde{C}_{0,\hbar}} (W^{-d/2} + B_\hbar(W)) e^{W+A_\hbar(W)} dW \\ &= 2^{1-d}\pi^{-d/2}\hbar^{\Delta(\alpha,d)}s^{d/2-1} \cdot \frac{i}{2\pi} \int_{\tilde{C}_0} e^W W^{-d/2} dW \cdot (1 + O(\hbar^{2-3\alpha}) + O(\hbar^{1-\alpha})). \end{aligned}$$

where in the last step we used the Trimming Lemma 3.2, and \tilde{C}_0 is a contour running from $-\infty - i0$ counterclockwise around $(-\infty, 0]$ to $-\infty + i0$. To see that the lemma is applicable, we note that

$$A_\hbar(W) = \hbar^{-1} \sum_{j \geq 3 \text{ odd}} (W\hbar^{1-\alpha}s^{-1})^j/j$$

Since there exists w_0 small enough such that if $|w| < w_0$, we have $\sum_{k \geq 1} |w|^{2k+1}/(2k+1) < 2|w|^3/3$. And since $|w| < \hbar^{1-\alpha-\epsilon}$, we can always achieve $w < w_0$ by letting \hbar being small enough. Thus for \hbar small enough, for $|W| < \hbar^{-\epsilon}$ we have

$$|A_\hbar(W)| \leq \hbar^{-1}(2/3)|W\hbar^{1-\alpha}/s|^3 = (2/3)|s|^{-3}\hbar^{2-3\alpha}|W|^3$$

Similarly, we have $|B_\hbar(W)| \leq C\hbar^{1-\alpha}$. This shows we can apply Lemma 3.2 with $a = \min(2-3\alpha, 1-\alpha)$. Finally, using Hankel's representation for the reciprocal of the Gamma function

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_{\tilde{C}_0} W^{-z} e^W dW,$$

we find

$$I_0 = \frac{2^{1-d}\pi^{-d/2}s^{d/2-1}}{\Gamma\left(\frac{d}{2}\right)} \hbar^{\Delta(\alpha,d)} (1 + O(\hbar^{2-3\alpha}) + O(\hbar^{1-\alpha})).$$

This completes the proof of Lemma 6.4. \square

Comparing the contribution from the critical point z_\pm and the pole singularity at $z = 1$,

$$I_\pm/I_0 = C\hbar^{\frac{d-1}{2}(1-3\alpha/2)}(1 + O(\hbar^{1-(3/2)\alpha}) + O(\hbar^{\alpha/2})) \quad (55)$$

we see for $\alpha < 2/3$, I_0 dominates. This concludes the $\Pi_\hbar(x, x)$ estimates claimed in Proposition 6.1.

The estimate for $\partial_{x_i}\Pi_\hbar(x, x)$ and $\partial_{x_i}\partial_{y_j}\Pi_\hbar(x, x)$ are similar, the singularity at $z = 1$ dominates, and each additional $(1-z)$ factor contribute an $\hbar^{1-\alpha}/s$ factor. We get

$$\begin{aligned} \Pi_\hbar^{-1}\partial_{x_i}\Pi_\hbar(x, x) &= -\left(\frac{d-2}{2}\right)x_i\hbar^{-\alpha}s^{-1}(1 + O(\hbar^{1-(3/2)\alpha}) + O(\hbar^{\alpha/2})) \\ \Pi_\hbar^{-1}\partial_{x_i}\partial_{y_j}\Pi_\hbar(x, x) &= \left(\left(\frac{d}{2}-1\right)\left(\frac{d}{2}-2\right)x_ix_js^{-2}\hbar^{-2\alpha} + d^{-1}\delta_{ij}\hbar^{-2+\alpha}s\right)(1 + O(\hbar^{1-(3/2)\alpha}) + O(\hbar^{\alpha/2})) \\ \Omega_{ij} &= \left(\frac{2-d}{2}x_ix_js^{-2}\hbar^{-2\alpha} + d^{-1}\delta_{ij}\hbar^{-2+\alpha}s\right)(1 + O(\hbar^{1-(3/2)\alpha}) + O(\hbar^{\alpha/2})) \end{aligned}$$

As long as $\alpha < 2/3$, the δ_{ij} term from Ω_{ij} dominates and we have

$$\Omega_{ij} = d^{-1} \delta_{ij} \hbar^{-2+\alpha} s (1 + O(\hbar^{1-(3/2)\alpha}) + O(\hbar^{\alpha/2})).$$

This completes the proof of Proposition 6.1. \square

Applying the Kac-Rice formula, we complete the proof of the allowed region part of the Theorem 1.5.

6.2. The forbidden annuli region.

PROPOSITION 6.5. *Let $x^2 = 1 + \hbar^\alpha s$ for any $s > 0, \alpha \in (0, 2/3)$, then*

$$\Pi_\hbar(x, x) = 2^{-d} \pi^{-\frac{d+1}{2}} \hbar^{-\frac{d-1}{2} - \alpha \frac{d+1}{4}} s^{-1/4-d/4} e^{-2/3 s^{3/2} \hbar^{-1+3\alpha/2}} (1 + O(\hbar^{\alpha/2}) + O(\hbar^{1-(3/2)\alpha})).$$

Moreover,

$$\Omega_{ij}(x) = \hbar^{-1-\alpha/2} (-x_i x_j + \delta_{ij}) (2s^{1/2})^{-1} (1 + O(\hbar^{1-(3/2)\alpha}) + O(\hbar^{\alpha/2})).$$

The implied constants in the estimate above are uniform for s in a compact subset of $(0, \infty)$.

Proof. As in §6.1, the initial contour $C_\epsilon = \{|z|^2 = 1 - \epsilon\}$ can be deformed freely inside S (defined in (29)). The relevant critical point of Φ is

$$z_c = 1 + 2(x^2 - 1) - 2\sqrt{(x^2 - 1)x^2} = 1 - 2\sqrt{\hbar^\alpha s} + O(\hbar^\alpha),$$

with

$$\Phi(z_c) = -\frac{2}{3} s^{3/2} \hbar^{3\alpha/2} + O(\hbar^{5\alpha/2}), \quad \Phi''(z_c) = \frac{\hbar^{\alpha/2} \sqrt{s}}{2} + O(\hbar^\alpha).$$

We may therefore deform C_ϵ in a $\hbar^{\alpha/2}$ neighborhood of z_c so that it is parametrized by

$$z_c + \hbar^{\alpha/2} i\eta, \quad \eta \in \mathbb{R}.$$

Then a similar stationary phase computation as in the Proposition 6.1 carries through to get the result for $\Pi_\hbar(x, x)$.

Next, we evaluate $\Omega_{ij}(x)$ using (56). Here we write Ω as

$$\Omega_{ij}(x) = \left\langle \frac{x_i x_j}{\hbar^2} \frac{2(z-w)^2}{(1+z)^2(1+w)^2} + \frac{\delta_{ij}}{\hbar} \frac{2z}{1-z^2} \right\rangle_{z,w} \quad (56)$$

where

$$\langle \cdots \rangle_{z,w} := \frac{\oint_z \oint_w (\cdots) A(z) A(w) e^{\Phi(z)/\hbar} e^{\Phi(w)/\hbar} dz dw}{\oint_z \oint_w A(z) A(w) e^{\Phi(z)/\hbar} e^{\Phi(w)/\hbar} dz dw}.$$

Applying the standard stationary phase method, we get

$$\langle f(z, w) \rangle_{z,w} = f(z_c, w_c) - \hbar^{1-\alpha/2} s^{-1/2} (f_{zz} + f_{ww})|_{(z_c, w_c)} (1 + O(\hbar^{1-\alpha/2}) + O(\hbar^\alpha))$$

Indeed, in computing the next order term in $\langle f(z, w) \rangle_{z,w}$, one need not worry about the pairing between $A(z)$ and $A(w)$ among themselves, since the same contribution from the

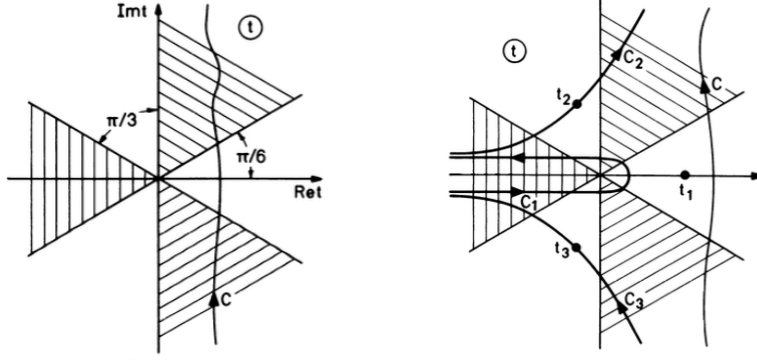


FIGURE 3. Defining contour for the generalized Airy function $\text{Ai}_k(u)$ (left panel), and its deformation when $x \ll 0$. (Taken from [BPD].)

denominator will cancel them out. Thus

$$\begin{aligned}
 \left\langle \frac{x_i x_j}{\hbar^2} \frac{2(z-w)^2}{(1+z)^2(1+w)^2} \right\rangle_{z,w} &= 0 + \left(\frac{x_i x_j}{\hbar^2} \frac{-\hbar^{1-\alpha/2} s^{-1/2}}{2} \right) (1 + O(\hbar^{1-\alpha/2}) + O(\hbar^\alpha)) \\
 \left\langle \frac{\delta_{ij}}{\hbar} \frac{2z}{1-z^2} \right\rangle_{z,w} &= \frac{\delta_{ij}}{\hbar} \frac{1}{2\hbar^{\alpha/2} \sqrt{s}} + O(\hbar^{\alpha/2}) + O(\hbar^{1-(3/2)\alpha}) \\
 \Omega_{ij}(u) &= \left\langle \frac{x_i x_j}{\hbar^2} \frac{2(z-w)^2}{(1+z)^2(1+w)^2} + \frac{\delta_{ij}}{\hbar} \frac{2z}{1-z^2} \right\rangle_{z,w} \\
 &= \hbar^{-1-\alpha/2} (-x_i x_j + \delta_{ij}) (2\sqrt{s})^{-1} (1 + O(\hbar^{1-\alpha(3/2)}) + O(\hbar^{\alpha/2}))
 \end{aligned}$$

□

Applying the Kac-Rice formula, we complete the proof of the forbidden region part of the Theorem 1.5.

APPENDIX A. THE WEIGHTED AIRY FUNCTIONS

For any $k \in \mathbb{R}$, we define the weighted Airy function $\text{Ai}_k(s)$ by

$$\text{Ai}_k(s) := \int_{\mathcal{C}} T^k \exp\left(\frac{T^3}{3} - Ts\right) \frac{dT}{2\pi i}, \quad s \in \mathbb{R} \quad (57)$$

The contour \mathcal{C} is coming from $e^{-i\theta_-}\infty$ and ending at $e^{i\theta_+}\infty$, for $\theta_- \in [-\pi/2, -\pi/6]$ and $\theta_+ \in [\pi/6, \pi/2]$, and stays within the right half plane ($\text{Re } T > 0$) (shown as C in Figure 3).

If $k = 0$, $\text{Ai}_0(s) = \text{Ai}(s)$. For positive integral weight, we have

$$\text{Ai}_k(s) = \left(-\frac{d}{dx}\right)^k \text{Ai}(s) \quad k \in \mathbb{Z}_+ \quad (58)$$

For negative real weight, we can use

$$T^{-k} = \frac{1}{\Gamma(k)} \int_0^\infty \rho^{k-1} e^{-\rho T} d\rho, \quad \text{Re}(T) > 0, k > 0$$

to get

$$\begin{aligned}\text{Ai}_{-k}(s) &= \frac{1}{\Gamma(k)} \int_{\mathcal{C}} \int_0^\infty \exp\left(\frac{T^3}{3} - T(s + \rho)\right) \rho^{k-1} d\rho \frac{dT}{2\pi i} \\ &= \frac{1}{\Gamma(k)} \int_0^\infty \text{Ai}(s + \rho) \rho^{k-1} d\rho\end{aligned}\quad (59)$$

The weighted Airy function has been considered before, see [BPD], there $W(z, n) = \Gamma(n + 1) \text{Ai}_{-(n+1)}(z)$. Here we quote the weighted Airy functions' asymptotic expansion from the above paper, cf Eq. (17), (18), (19) there.

PROPOSITION A.1 ([BPD]). *Fix any $k \in \mathbb{R}$.*

(1) *For $s \gg 0$, we get*

$$\text{Ai}_{-k}(s) = \frac{1}{2\sqrt{\pi}} \frac{e^{-2/3s^{3/2}}}{s^{(2k+1)/4}} \left[1 - \frac{k^2 + 2k}{4s^{3/2}} + O(s^{-3})\right].$$

(2) *For $s \ll 0$, we get*

$$\text{Ai}_{-k}(s) = \sum_{j=0}^{\infty} \frac{|s|^{k-3j-1}}{3^j \Gamma(k-3j)} + \frac{\sin(2|s|^{2/3}/3 - (2k-1)\pi/4)}{\pi^{1/2}|s|^{(2k+1)/4}} [1 + O(|s|^{-1})]$$

In first term summation, if j is such that $k-3j$ is a non-positive integer, then $|\Gamma(k-3j)| = \infty$ and the corresponding term vanishes.

The only difference with [BPD] is that for $s \gg 0$ case, we computed the second order term, and we removed the $k \geq 1$ condition. Since the arguments in that paper still holds verbatim, we will not give the detail of the proof here.

The following lemma is used in the proof of Theorem 1.1.

LEMMA A.2 (Product formula for Airy function).

$$\text{Ai}(x) \text{Ai}(y) = \int_{\mathcal{C}} e^{\frac{2T^3}{3} - (x+y)T - \frac{(x-y)^2}{8T}} \sqrt{\frac{1}{2\pi T}} \frac{dT}{(2\pi i)}$$

In particular, if $x = y$, we get

$$\text{Ai}(x)^2 = 2^{-1/6} (2\pi)^{-1/2} \text{Ai}_{-1/2}(2^{2/3}x).$$

Proof. From the integral expression of airy function we obtain

$$\text{Ai}(x) \text{Ai}(y) = \int_{T_1 \in \mathcal{C}} \int_{T_2 \in \mathcal{C}} e^{\frac{T_1^3 + T_2^3}{3} - xT_1 - yT_2} \frac{dT_1}{2\pi i} \frac{dT_2}{2\pi i}.$$

If we straighten \mathcal{C} to be $i\mathbb{R} + \epsilon$ for some $\epsilon > 0$, then we can reparametrize the integration variables as $T_1 = T + S/2$, $T_2 = T - S/2$, for $T \in i\mathbb{R} + \epsilon$ and $S \in i\mathbb{R}$. The integration becomes

$$\begin{aligned}\text{Ai}(x) \text{Ai}(y) &= \int_{T \in i\mathbb{R} + \epsilon} \int_{S \in i\mathbb{R}} e^{\frac{2T^3}{3} - (x+y)T + \frac{1}{2}(S^2T - S(x-y))} \frac{dT dS}{(2\pi i)^2} \\ &= \int_{\mathcal{C}} e^{\frac{2T^3}{3} - (x+y)T - \frac{(x-y)^2}{8T}} \sqrt{\frac{1}{2\pi T}} \frac{dT}{(2\pi i)}\end{aligned}$$

□

APPENDIX B. ANALYTIC STATIONARY PHASE METHOD

We recall here the version of the method of stationary phase that we will use.

PROPOSITION B.1 (Thm. 7.7.5 Vol. 3 [Hor]). *Let $K \subseteq \mathbb{R}^n$ be a compact set, X and open neighborhood of K and k be a positive integer. If $u \in C_0^{2k}(K)$, $f \in C^{3k+1}(X)$ and $\text{Im}(f) \geq 0$ in X , $\text{Im}(f(x_0)) = 0$, $f'(x_0) = 0$, $\det f''(x_0) \neq 0$ and $f' \neq 0$ on $K \setminus \{x_0\}$, then*

$$\left| \int_X u(x) e^{i\omega f(x)} dx - e^{i\omega f(x_0)} \cdot \det(\omega f''(x_0)/2\pi i)^{-1/2} \sum_{j < k} \omega^{-j} L_j u \right| \leq C \omega^{-k} \cdot \det(\omega f''(x_0)/2\pi i)^{-1/2} \sum_{|\alpha| \leq 2k} \sup |D^\alpha u|.$$

The constant C is uniform over any bounded set S in C^{3k+1} as long as $|\det f''(x)|$ is uniformly bounded away from 0 for all $f \in S$.

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